Radiation condition at infinity for the high-frequency Helmholtz equation: optimality of a non-refocusing criterion

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Abstract

We consider the high frequency Helmholtz equation with a variable refraction index $n^2(x)$ ($x \in \mathbb{R}^d$), supplemented with a given high frequency source term supported near the origin $x = 0$. A small absorption parameter $\alpha_\varepsilon > 0$ is added, which somehow prescribes a radiation condition at infinity for the considered Helmholtz equation. The semi-classical parameter is $\varepsilon > 0$. We let ε and α_ε go to zero *simultaneaously*. We study the question whether the indirectly prescribed radiation condition at infinity is satisfied *uniformly* along the asymptotic process $\varepsilon \to 0$, or, in other words, whether the conveniently rescaled solution to the considered equation goes to the *outgoing* solution to the natural limiting Helmholtz equation.

This question has been previously studied by the first autor in [4]. In [4], it is proved that the radiation condition is indeed satisfied uniformly in ε, provided the refraction index satisfies a specific *non-refocusing condition*, a condition that is first pointed out in this reference. The non-refocusing condition requires, in essence, that the rays of geometric optics naturally associated with the high-frequency Helmholtz operator, and that are sent from the origin $x = 0$ at time $t = 0$, should not refocus at some later time $t > 0$ near the origin again.

In the present text we show the *optimality* of the above mentionned non-refocusing condition, in the following sense. We exhibit a refraction index which *does* refocus the rays of geometric optics sent from the origin near the origin again, and, on the other hand, we completely compute the asymptotic behaviour of the solution to the associated Helmholtz equation: we show that the limiting solution *does not* satisfy the natural radiation condition at infinity. More precisely, we show that the limiting solution is a *perturbation* of the outgoing solution to the natural limiting Helmholtz equation, and that the perturbing term explicitly involves the contribution of the rays radiated from the origin which go back to the origin. This term is also conveniently modulated by a phase factor, which turns out to be the action along the above rays of the hamiltonian associated with the semiclassical Helmholtz equation.

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1 Introduction

1.1 General introduction

In this article, we study the convergence as ε approaches 0 of w^ε, solution to the following rescaled Helmholtz equation

$$i\varepsilon\alpha_\varepsilon\, w_\varepsilon(x) + \frac{\Delta_x}{2} w_\varepsilon(x) + n^2(\varepsilon x) w_\varepsilon(x) = S(x), \quad x \in \mathbb{R}^d \quad (d \geq 3). \tag{1}$$

Here α_ε is an absorption parameter, $n^2(x)$ is a space-dependent refraction index[1] and $S(x)$ is a given and smooth source term. In the sequel, we assume the following:

- The absorption parameter α_ε satisfies[2]

$$\alpha_\varepsilon > 0, \qquad \alpha_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

- The smooth refraction index $n^2(x) \in C^\infty(\mathbb{R}^d)$ is a possibly long-range perturbation of a positive constant $n_\infty^2 > 0$ at infinity, namely, for some $\rho > 0$, we have

$$\forall\, \alpha \in \mathbb{N}^d, \quad \exists C_\alpha, \quad \forall\, x \in \mathbb{R}^d, \quad \left|\partial_x^\alpha \left(n^2(x) - n_\infty^2\right)\right| \leq C_\alpha \langle x\rangle^{-\rho-\alpha}, \tag{2}$$

  where we denote as usual $\langle x\rangle := (1 + |x|^2)^{1/2}$.

[1]Here and below we use the standard notation $n^2(x)$, a squared term, assuming in doing so that the corresponding term is everywhere non-negative. This is a harmless abuse of notation, since the refraction index $n^2(x)$ that is eventually chosen in our analysis is negative for certain values of x. The reader may safely skip this fact, since the Helmholtz equation also arises in the spectral analysis of Schrödinger operators, where the refraction index becomes $E - V(x)$ where E is an energy and $V(x)$ is a space-dependent potential, and the term $E - V(x)$ may change sign in that context.

[2]The limiting case $\alpha_\varepsilon = 0^+$ can be considered along our analysis, see below.

- The source term $S(x)$ belongs to the Schwartz class[3] $\mathcal{S}(\mathbb{R}^d)$.

The question we raise is the following. Thanks to the absorption parameter $\alpha_\varepsilon > 0$ in (1), the sequence of solutions w_ε is uniquely defined (see below for the limiting case $\alpha_\varepsilon = 0^+$). On top of that, and as a consequence of specific homogeneous bounds obtained by Perthame and Vega in [14] (see [5] for extensions by Jecko and the first author, as well as [6]), it is clear that the sequence w_ε is bounded in some weighted L^2 space, uniformly in ε. Hence the sequence w_ε possesses a limit (up to subsequences), say in the distribution sense, and the limit $w = \lim w_\varepsilon$ satisifies in the distribution sense the Helmholtz equation

$$\frac{\Delta_x}{2}\, w + n^2(0)\, w = S, \tag{3}$$

where the variable coefficients refraction index $n^2(\varepsilon x)$ in (1) has now coefficients frozen at the origin $x = 0$.

Now, the difficulty is, the Helmholtz equation (3) does not have a uniquely defined solution. At least two distinct solutions exist, namely the outgoing solution, defined as

$$w_{out}(x) := \lim_{\delta \to 0^+} \left(i\delta + \frac{\Delta_x}{2} + n^2(0) \right)^{-1} S(x), \tag{4}$$

and the incoming solution, defined similarly as $w_{in} = \lim\limits_{\delta \to 0^+} \left(-i\delta + \dfrac{\Delta_x}{2} + n^2(0) \right)^{-1} S$. Equivalently, the outgoing solution may be defined as the unique solution to the Helmholtz equation (3) which satisfies the so-called Sommerfeld radiation condition at infinity, namely

$$\frac{x}{|x|} . \nabla_x w_{out}(x) + i\sqrt{2}\, n(0)\, w_{out}(x) = O\left(\frac{1}{|x|^2}\right), \qquad \text{as } |x| \longrightarrow +\infty. \tag{5}$$

This formulation means that w_{out} is required to oscillate like $w_{out} \sim \exp\left(-i\sqrt{2}\, n(0)|x|\right)/|x|$ as $|x| \to \infty$. Similarly, the incoming solution satisfies the following radiation condition at infinity, namely $(x/|x|) . \nabla_x w_{in} - i\sqrt{2}\, n(0)\, w_{in} = O\left(1/|x|^2\right)$, meaning that $w_{in} \sim \exp\left(+i\sqrt{2}\, n(0)|x|\right)/|x|$ as $x \to \infty$.

In that perspective, and due to the positive absorption parameter $\alpha_\varepsilon > 0$ in (1), it is natural to *expect* that the previously defined sequence w_ε goes to the *outgoing* solution w_{out} to (3).
This is the question we address here.

It turns out that delicate analytical tools are needed to provide a clean understanding of the phenomena at hand, and to establish whether $w_\varepsilon \sim w_{out}$ as $\varepsilon \to 0$. The basic difficulty is a conflict between a local and a global phenomenon. On the one hand, the obvious fact that w_ε goes to a solution to (3) is *local*: locally in x, *i.e.* in the distribution sense, the variable refraction index $n^2(\varepsilon x)$ goes to the value $n^2(0)$ at the origin. On the other hand, the positive absorption parameter $\alpha_\varepsilon > 0$ in (1) somehow asserts that w_ε is an *outgoing* solution to $\Delta_x w_\varepsilon/2 + n^2(\varepsilon x)\, w_\varepsilon = S$, hence introducing the value at infinity $n_\infty = \lim\limits_{x \to \infty} n(\varepsilon x) = \lim\limits_{x \to \infty} n(x)$, the solution w_ε should roughly oscillate like $w_\varepsilon \sim \exp\left(-i\sqrt{2}\, n_\infty |x|\right)/|x|$ at infinity. This is a *global* phenomenon. Now, all this is to be compared with the fact that w_{out} oscillates like $w_{out} \sim \exp\left(-i\sqrt{2}\, n(0)|x|\right)/|x|$ at infinity. Due to the fact that $n_\infty \neq n(0)$, the radiation condition at infinity satisfied by w_ε for any positive value $\varepsilon > 0$ is *a priori* incompatible with the radiation condition at infinity satisfied by the expected limit w_{out}: the radiation condition at infinity cannot be followed at once uniformly

[3] This assumption may be considerably relaxed at the price of some irrelevant technicalities.

in ε, in any direct fashion (this is not in contradiction with the expected *local* convergence of w_ε towards w_{out}.)

Before going further, let us mention that the above question stems from a series of articles [1], [3] about the *high-frequency* Helmholtz equation (Equation (1) is a *low-frequency* equation) (see also [9] and [10] for similar considerations, in the case of a discontinuous refraction index, as well as [16] and [17] for the case of a variable absorption coefficient). These two papers investigate the high-frequency behaviour, in terms of semi-classical measures, of high-frequency Helmholtz equations of the form

$$i\varepsilon\alpha_\varepsilon u_\varepsilon(x) + \frac{\varepsilon^2}{2}\,\Delta_x u_\varepsilon(x) + n^2(x)u_\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} S\left(\frac{x}{\varepsilon}\right) \quad (x \in \mathbb{R}^d). \tag{6}$$

The link between the low-frequency equation (1) that is the purpose of this article, and the high-frequency equation (6) is provided by the following basic observation: the function w_ε satisfies (1) if and only if the rescaled function

$$u_\varepsilon(x) = \frac{1}{\varepsilon^{d/2}}\, w_\varepsilon\left(\frac{x}{\varepsilon}\right) \tag{7}$$

satisfies (6). In that picture, the main phenomenon to be described in (6) is the possibility of *resonances* between the high-frequency waves selected by the Helmholtz operator $\varepsilon^2\Delta_x/2+n^2(x)$, and the high-frequency waves carried by the rescaled source term $\varepsilon^{-d/2}\,S(x/\varepsilon)$, both having the same wavelength ε. Amongst others, it is established in [1], [3] that the semiclassical measure associated with u_ε can be completely computed provided w_ε indeed converges towards w_{out}, this latter requirement being left as a conjecture in the cited papers. This is the motivation for the question we address here.

In [4], the first positive convergence result $w_\varepsilon \to w_{out}$ is established. This results requires, amongst others, a specific and original *non-refocusing condition* on the refraction index $n^2(x)$ (called "transversality condition" in the original paper). This condition (see below for details) roughly asserts that the rays of geometric optics associated with the the semi-classical Helmholtz operator $\varepsilon^2\Delta_x/2 + n^2(x)$ cannot focus at some positive time $t > 0$ near the origin $x = 0$ when issued from the origin at time $t = 0$. Later, X.-P. Wang and P. Zhang [19] proved a similar, positive result, using a so-called virial assumption which is stronger than the above non-refocusing condition. J.-F. Bony in [2] establishes along quite different lines a positive result that is similar in spirit, requiring a weaker non-refocusing condition.

The goal of the present text is to prove in some sense the *optimality* of the non-refocusing condition pointed out in [4].

We construct a refraction index $n^2(x)$ which violates the non-refocusing condition (rays of geometric optics issued from the origin do refocus close to the origin at some later time), and, by explicitly computing the asymptotic behaviour of w_ε thanks to an appropriate amplitude/phase representation developped in [4], we prove that

$$w_\varepsilon \underset{\varepsilon\to 0}{\sim} w_{out} + \underbrace{\text{perturbation}}_{\neq 0},$$

where the perturbation is computed as well. It explicitly involves the contribution of the rays issued from the origin which go back to the origin at some positive time, modulated by a phase factor that is the action, along these rays, of the hamiltonian associated with the high-frequency Helmholtz operator.

1.2 The non-refocusing condition

As already mentionned, the asymptotic behaviour of w_ε is dictated by that of the rescaled function $u_\varepsilon(x) = \varepsilon^{-d/2} w_\varepsilon(x/\varepsilon)$. The function u_ε is w_ε rescaled at the semi-classical scale, see (6) and (7). This is translated by the following identity, valid for any smooth test function $\phi \in \mathcal{S}(\mathbb{R}^d)$, namely

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \qquad \langle w_\varepsilon, \phi \rangle = \left\langle u_\varepsilon, \frac{1}{\varepsilon^{d/2}} \phi\left(\frac{x}{\varepsilon}\right) \right\rangle.$$

where we denote as usual $\langle w_\varepsilon, \phi \rangle := \int_{\mathbb{R}^d} w_\varepsilon(x)\, \phi^*(x)\, dx$, and $*$ denotes complex conjugation. In other words, the weak limit $\langle w_\varepsilon, \phi \rangle$ of w_ε can be computed as the weak limit *at the semi-classical scale* of u^ε, namely the limit of $\langle u_\varepsilon, \varepsilon^{-d/2}\phi(x/\varepsilon)\rangle$. This first observation is the main reason why semi-classical tools play a key role in our analysis.

Besides, the asymptotic study of (1) is done here by transforming the problem into a time-dependent problem. This approach, introduced in [4], has been used since by J.F.-Bony ([2]) to study the Wigner measure associated to (6), or by J. Royer ([16]) when the absorption α_ε depends on x. It consists in writing the solution w_ε as the integral over the whole time of the propagator associated with $i\varepsilon\alpha_\varepsilon + \Delta_x/2 + n^2(\varepsilon\, x)$, namely

$$w_\varepsilon(x) = i \int_0^{+\infty} e^{-\alpha_\varepsilon t} e^{it\left(\frac{\Delta_x}{2} + n^2(\varepsilon\, x)\right)}\, S(x)\, dt. \tag{8}$$

In the same way the outgoing solution can be written as

$$w_{out}(x) := i \int_0^{+\infty} e^{it\left(\frac{\Delta_x}{2} + n^2(0)\right)}\, S(x)\, dt.$$

In that picture, proving or disproving the convergence $w_\varepsilon \sim w_{out}$ reduces to passing to the limit in the above time integral.

Combining the two above observations, the basic first step of our analysis consists in writing, for any given test function ϕ, an in [4],

$$\begin{aligned}\langle w_\varepsilon\, ,\, \phi \rangle &= \left\langle u^\varepsilon\, ,\, \varepsilon^{-d/2}\phi(x/\varepsilon)\right\rangle \\ &= \frac{i}{\varepsilon} \int_0^{+\infty} e^{-\alpha_\varepsilon\, t} \left\langle U_\varepsilon(t)\, S_\varepsilon\, ,\, \phi_\varepsilon \right\rangle dt,\end{aligned} \tag{9}$$

where we use the notation

$$S_\varepsilon(x) := \frac{1}{\varepsilon^{d/2}}\, S\left(\frac{x}{\varepsilon}\right), \qquad \text{and similarly } \phi_\varepsilon(x) := \frac{1}{\varepsilon^{d/2}}\, \phi\left(\frac{x}{\varepsilon}\right), \tag{10}$$

where the semi-classical propagator associated with the semi-classical Hamiltonian $\varepsilon^2\Delta_x/2 + n^2(x)$ is

$$U_\varepsilon(t) = \exp\left(i\, \frac{t}{\varepsilon} \left(\frac{\varepsilon^2}{2}\Delta_x + n^2(x)\right)\right) \tag{11}$$

It is fairly clear on formula (9) that the asymptotics $\varepsilon \to 0$ in $\langle w_\varepsilon, \phi \rangle$ is dominated on the one hand by the concentration of the rescaled test function ϕ_ε close to the origin at the semi-classical scale ε, and on the other hand by the oscillations induced by the semi-classical propagator $U^\varepsilon(t)$ at the semi-classical scale ε as well. The point is to measure the possible constructive intereference between both waves.

As standard in semiclassical analysis we define the semiclassical symbol

$$h(x, \xi) = \frac{|\xi|^2}{2} - n^2(x), \tag{12}$$

associated with the semiclassical Schrödinger operator $-\frac{\varepsilon^2}{2}\Delta_x - n^2(x)$. The semi-classical propagator $U_\varepsilon(t)$ is known to roughly propagate the information along the rays of geometric optics, defined as the solutions to the Hamiltonian ODE associated with h, namely (see *e.g.* [8], [13], or [15])

$$\begin{cases} \dfrac{\partial}{\partial t} X(t, x, \xi) = \Xi(t, x, \xi), & X(0, x, \xi) = x, \\ \dfrac{\partial}{\partial t} \Xi(t, x, \xi) = \nabla_x n^2(X(t, x, \xi)), & \Xi(0, x, \xi) = \xi. \end{cases} \tag{13}$$

It is clear as well that the integral $\int_0^{+\infty} \dots$ in (9) carries most of its energy, semi-classically, over the zero energy level of h, defined as

$$H_0 := \left\{ (x, \xi) \in \mathbb{R}^{2d}, \text{ s.t. } h(x, \xi) = 0 \right\}. \tag{14}$$

In view of the integral (9) and of the above considerations, the following definitions are natural. The first definition is standard.

Definition 1.1. [non-trapping condition]
The refraction index n^2 is said non-trapping on the zero energy level *whenever for each $(x, \xi) \in H_0$, the associated trajectory $(X(t, x, \xi), \Xi(t, x, \xi))$ satisfies*

$$\lim_{t \to +\infty} |X(t, x, \xi)| = +\infty.$$

When the refraction index is non-trapping, the rough idea is that any trajectory $X(t, x, \xi)$ on the zero energy level leaves any given neighbourhood of the origin $x = 0$ in finite time, making the above integral $\int_0^{+\infty} \dots$ in (9) converge with respect to the bound $t = +\infty$.

The second definition comes from [4] (this assumption is called "transversality condition" in the original text).

Definition 1.2. [non-refocusing condition]
We say that n^2 satisfies the non-refocusing condition *if the refocusing set, defined as*

$$M := \left\{ (t, \xi, \eta) \in]0, +\infty[\times \mathbb{R}^{2d} \ \ s.t. \ \frac{|\eta|^2}{2} = n^2(0), \ X(t, 0, \xi) = 0, \ \Xi(t, 0, \xi) = \eta \right\} \tag{15}$$

is such that M is a submanifold of $]0, +\infty[\times \mathbb{R}^{2d}$ and M satisfies

$$dim\, M < d - 1.$$

When the non-refocusing condition is satisfied, the rough idea is that the trajectories $X(t, 0, \xi)$ on the zero energy level issued from the origin $x = 0$ at time $t = 0$ cannot accumulate in any given neighbourhood of the origin $x = 0$ at later times $t > 0$ (this is encoded in the requirement on $\dim M$). Technically speaking, an appropriate stationary phase argument in formula (9) allows to exploit in [4] the non-refocusing condition and to prove the weak convergence of w_ε towards w_{out} under this assumption. The main result in [4] is the following: when the refraction index is both non-trapping *and* satisfies the above non-refocusing condition, then $w_\varepsilon \sim w_{out}$ as $\varepsilon \to 0$ weakly.

Recently, J.F. Bony in [2] shows the convergence of the *Wigner measure* associated with w_ε. He requires a geometrical assumption on the index of refraction that is in the similar spirit, yet weaker, than the above non-refocusing condition, namely

$$\text{meas}_{n-1}\left\{\xi \in \sqrt{2n^2(0)}\,\mathbb{S}^{d-1};\quad \exists\, t>0 \quad X(t,0,\xi)=0\right\}=0, \tag{16}$$

where meas_{n-1} is the Euclidian surface measure on $\sqrt{2n^2(0)}\,\mathbb{S}^{d-1}$ and $\mathbb{S}^{d-1}$ denotes the unit sphere in dimension d. Besides, inspired by [4], he constructs a refraction index which is both non-trapping and does not satisfy condition (16), and in that case he proves the *non-uniqueness* of the limiting of the Wigner measure.

The goal of this paper is to construct a refraction index that is both non-trapping and violates the non-refocusing condition, and to establish in that case that w_ε goes weakly to a function of the form "w_{out}+perturbation", for some explicitly computed and non-zero perturbation. To be more accurate, we construct below a refraction index for which the above refocusing manifold $M=\left\{(t,\xi,\eta) \text{ s.t. } \frac{|\eta|^2}{2}=n^2(0),\ X(t,0,\xi)=0,\ \Xi(t,0,\xi)=\eta\right\}$ is smooth, yet has dimension $\dim M = d-1$, a critical case, and we prove $w_\varepsilon \sim$ "w_{out}+perturbation" in that situation.

1.3 Construction of the refraction index and statement of our main result

Let us first examine the case of dimension $d=2$. Let M_s be a circular mirror centered at the origin. Any standard ray issued from the origin $x=0$ hits the mirror and goes back to the origin at some later time: refocusing occurs in a strong fashion. However all rays are trapped inside the circular mirror, leading to a trapping situation, in the sense of definition 1.1. To recover a non-trapping and refocusing situation, it is necessary to consider an angular aperture of the circular mirror, with total aperture $<\pi$. This is shown in figure 1: the circular mirror with total aperture $<\pi$ provides a (non-smooth) non-trapping and refocusing refraction index. To

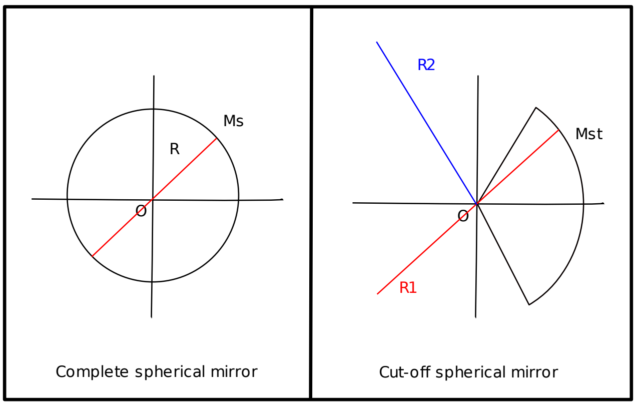

Figure 1: Spherical mirror in dimension 2

transform the above paradigm into a smooth one, some regularizations need to be performed. The construction needs to be done in any dimension $d \geq 2$ as well.

Let us first introduce the hyperspherical coordinates $(r, \theta_1, \ldots, \theta_{d-1})$ in dimension $d \geq 2$

$$\begin{aligned} x_1 &= r\cos(\theta_1), \\ x_2 &= r\sin(\theta_1)\cos(\theta_2), \\ x_3 &= r\sin(\theta_1)\sin(\theta_2)\cos(\theta_3), \\ &\vdots \\ x_{d-1} &= r\sin(\theta_1)\ldots\sin(\theta_{d-2})\cos(\theta_{d-1}), \\ x_d &= r\sin(\theta_1)\ldots\sin(\theta_{d-2})\sin(\theta_{d-1}), \end{aligned}$$

with

$$\theta_1 \in [0,\pi], \quad \theta_j \in [0,2\pi] \text{ whenever } j \geq 2 \text{ when } d \geq 3, \qquad \text{and } \theta_1 \in [-\pi,\pi] \text{ when } d = 2.$$

Next, we choose a *fixed*, smooth cut-off function χ on $\mathbb{R}$ such that

$$\chi(t) = 1, \quad \forall\ |t| \leq 1, \qquad \chi(t) = 0, \quad \forall\ |t| \geq 2, \qquad \chi(t) \geq 0, \quad \forall t \in \mathbb{R}. \tag{17}$$

We choose a radius $R > 0$ and define the radial function

$$f(x) \equiv f(r) := \chi\left(2(r-R)\right), \quad \forall\ x = (r, \theta_1, \ldots, \theta_{d-1}). \tag{18}$$

We choose an angle (aperture) $\theta_0 \in [0, \pi/4[$, and define the angular function

$$g(x) \equiv g(\theta_1) := \chi\left(\frac{\theta_1}{\theta_0}\right), \quad \forall\ x = (r, \theta_1, \ldots, \theta_{d-1}). \tag{19}$$

a smooth version of the angular aperture $|\theta_1| \leq \theta_0$. Finally, we choose two parameters $n_\infty^2 > 0$ and $\lambda > 0$ such that

$$n_\infty^2 < \lambda. \tag{20}$$

We introduce the following

Definition 1.3. [refraction index]
We define the refraction index, retained in the whole subsequent analysis, as the following smooth version of the circular mirror with total aperture $\theta_0 < \pi/4$, namely[4]

$$n^2(x) := n_\infty^2 - \lambda f(x) g(x) \equiv n_\infty^2 - \lambda f(r) g(\theta_1), \qquad \forall x \in \mathbb{R}. \tag{21}$$

We are now in position to state our main result. Let $(e_1, \ldots, e_d)$ be the canonical basis of $\mathbb{R}^d$. Since the direction e_1 is a symmetry axis for our refraction index, we introduce for later purposes the space $M_d(\mathbb{R})$ of square matrices of dimension d, we denote by $\mathbb{O}_d(\mathbb{R})$ the space of orthogonal matrices, and we introduce the notation

$$\mathbb{O}_{d,1}(\mathbb{R}) := \left\{A \in \mathbb{O}_d(\mathbb{R}), \text{ s.t. } Ae_1 = e_1\right\}. \tag{22}$$

The refraction index $n^2(x)$ in (21) is invariant under the action of $\mathbb{O}_{d,1}(\mathbb{R})$. We last introduce a particular set of speeds, namely the set of initial speeds ξ such that the zero energy trajectory $X(t, 0, \xi)$ issued from the origin at time $t = 0$ is reflected towards the origin at some later time $t > 0$. With the retained value of $n^2(x)$, we arrive at the definition

[4]The refraction index is negative in a bounded region of x. As already mentioned, we still use the abuse of notation consisting in using the squared of n.

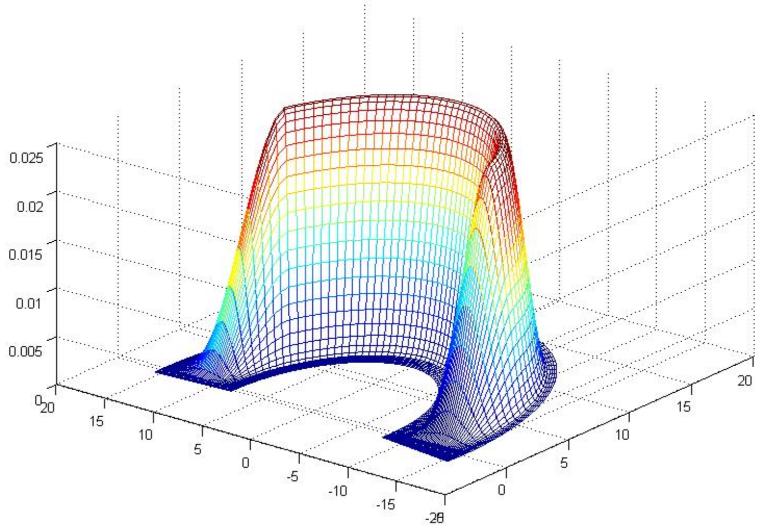

Figure 2: The function $n_\infty^2 - n^2(x) = \lambda\, f(x)\, g(x)$ in dimension $d = 2$

Definition 1.4. [reflected rays]
The reflection set I_{θ_0} *is defined as*

$$I_{\theta_0} = \left\{ \xi := (|\xi|, \theta_1, \ldots, \theta_{d-1}) \in \mathbb{R}^d \;\; s.t. \;\; \theta_1 \in [-\theta_0, +\theta_0] \;\; and \;\; |\xi| = \sqrt{2\, n^2(0)} \right\}.$$

Note that the (intuitive) fact that a velocity ξ is such that $X(t, 0, \xi)$ hits the origin at some time $t > 0$ if and only if $\xi \in I_{\theta_0}$, is proved later (see section 2.2).

Our main result in this text is the

Theorem 1.5. [Main Result]
Let n^2 *be the refraction index defined in (21). Assume the aperture* $\theta_0 < \pi/4$ *and the radius* $R > 0$ *satisfy the smallness condition*

$$1 - \cos(2\theta_0) < \frac{1}{2R}. \tag{23}$$

Assume $d \geq 3$*. Then, the following holds:*

i) The index n^2 *is non-trapping on the zero-energy level* $H_0 = \{(x, \xi) \;\; s.t. \;\; |\xi|^2/2 - n^2(x) = 0\}$.

ii) The refocusing set $M = \{(t, \xi, \eta) \;\; s.t. \;\; |\eta|^2 = 2n^2(0),\, X(t, 0, \xi) = 0,\, \Xi(t, 0, \xi) = \eta\}$ *(see (15)) is a smooth submanifold of* $]0, +\infty[\times\mathbb{R}^{2d}$*, with boundary, and its dimension has the critical value*

$$\dim(M) = d - 1.$$

iii) Assume the source term S *satisfies* $S \in \mathcal{S}(\mathbb{R}^d)$*. Then, we have*

$$\forall\, \phi \in \mathcal{S}(\mathbb{R}^d), \qquad \left\langle w_\varepsilon - (w_{out} + L_\varepsilon)\;,\; \phi \right\rangle \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

where the distribution L_ε is defined for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ through

$$\langle L_\varepsilon, \phi \rangle = \\ C_{n^2,d} \int_{I_{\theta_0}} \exp\left(\frac{i}{\varepsilon} \int_0^{T_R} \left(\frac{|\Xi(s,0,\xi)|^2}{2} + n^2(X(s,0,\xi)) \right) ds \right) \widehat{S}(\xi) \widehat{\phi^*}(-\xi)\, d\sigma_{\theta_0}(\xi). \tag{24}$$

Here $d\sigma_{\theta_0}$ denote the natural Euclidean surface measure on I_{θ_0} (see definition 1.4), the return time $T_R > 0$ is the unique time[5] *such that for any $\xi \in I_{\theta_0}$ we have $X(T_R, 0, \xi) = 0$, and the constant $C_{n^2,d} \neq 0$ can be explicitly computed and depends only on the index n^2 and on the dimension d.*

Remark. *The condition (23) is technical, and requires the aperture θ_0 to be small: it ensures the trajectories cannot be trapped by the refraction index.*

Remark. *Note in passing that the constraint $d \geq 3$, which is also needed in reference [4], comes from a stationary phase argument. This constraint on the dimension is standard in the analysis of Schrödinger-like operators. It comes from the fact that the dispersion induced by the free Schrödinger operator acts like $t^{-d/2}$, a factor that is integrable close to $t = +\infty$ whenever $d \geq 3$.*

Remark. *Let $\xi_0 := (\sqrt{2}\, n(0), 0, \ldots, 0)$. The distribution L_ε can as well be written as*

$$\langle L_\varepsilon\,,\, \phi \rangle = \\ C_{n^2,d} \exp\left(\frac{i}{\varepsilon} \int_0^{T_R} \left(\frac{|\Xi(s,0,\xi_0)|^2}{2} + n^2(X(s,0,\xi_0)) \right) ds \right) \left(\int_{I_{\theta_0}} \widehat{S}(\xi) \widehat{\phi^*}(-\xi)\, d\sigma_{\theta_0}(\xi) \right).$$

This formulation illustrates in a clearer way the fact that if the source S radiates towards the mirror, then w_ε converges towards a non-trivial perturbation of w_{out}.

Note in passing that in the present counter-example, as in the paper by J.-F. Bony [2] , only subsequences *of w_ε converge, due to the above oscillatory factor* $\exp(i\, \mathrm{const.}/\varepsilon)$.

Remark. *In the chosen hyperspherical coordinates, the Euclidean measure $d\sigma_{\theta_0}(\xi)$ coincides with $d\sigma_{\theta_0}(\xi) = n(0)^{d-1}\, d\sigma(\theta_1, \ldots, \theta_{d-1})$, where $d\sigma(\theta_1, \ldots, \theta_{d-1})$ denotes the standard euclidean surface measure on the unit sphere $\mathbb{S}^{d-1}$.*

1.4 Preliminary reduction of the proof

Our main result contains three distinct statements. Items (i) and (ii) are of geometric nature, and merely concern the behavious of the classical trajectories associated with the retained refraction index. Their proof is performed in sections 2.1 and 2.2, respectively. Item (iii) is the main item, and concerns the asymptotic analysis of w_ε. Since our analysis heavily relies on tools previously developped in [4], we briefly recall here some of these tools and indicate how the analysis of w_ε can be reduced to a simpler sub-problem. We postpone the analysis of the reduced subproblem, hence of item (iii) of our main result, to section 3 below.

As already indicated, given a smooth test function ϕ, we start from the formulation

$$\langle w_\varepsilon, \phi \rangle = \frac{i}{\varepsilon} \int_0^{+\infty} e^{-\alpha_\varepsilon t} \langle U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon \rangle\, dt.$$

[5] The fact that all these quantities exist and are well defined is part of the Theorem, and is proved in section 2.2.

(See above for the notation). The next step consists in splitting the above time integral into four time scales, namely very small, small, moderate, and large time scales. To do so, we take one small parameter $\theta > 0$ and two large parameters $T_0 > 0$ and $T_1 > 0$, and split the above time integral into the four zones

$$0 \leq t \leq T_0\,\varepsilon, \quad T_0\,\varepsilon \leq t \leq \theta, \quad \theta \leq t \leq T_1, \quad T_1 \leq t \leq +\infty \quad (\theta \ll 1, \quad T_0, T_1 \gg 1).$$

Technically, we use a smooth splitting, based on the already used cut-off function χ (see (17)). Besides, we also distinguish between the contribution of zero and non-zero energies, namely taking a small parameter $\delta > 0$, we write, in the sense of functional caculus for self-adjoint operators, the identity

$$1 = \chi_\delta(H_\varepsilon) + (1-\chi_\delta)(H_\varepsilon), \text{ where } H_\varepsilon := \frac{\varepsilon^2}{2}\Delta_x + n^2(x), \text{ and } \chi_\delta(s) := \chi\left(\frac{s}{\delta}\right) \ (s \in \mathbb{R}, \delta \ll 1).$$

The main intermediate result of the present subsection is the following

Proposition 1.6. [Main intermediate result]
Take a test function $\phi \in \mathcal{S}(\mathbb{R}^d)$. Define $\widetilde{w_\varepsilon}$ as

$$\langle \widetilde{w_\varepsilon}, \phi \rangle := \frac{i}{\varepsilon}\int_\theta^{T_1} (1-\chi)\left(\frac{t}{\theta}\right)\, e^{-\alpha_\varepsilon t}\left\langle U_\varepsilon(t)\,\chi_\delta(H_\varepsilon)\,S_\varepsilon, \phi_\varepsilon\right\rangle dt.$$

Then, there is a large $T_1 > 0$ such that for any small $\delta > 0$, and any small $\theta > 0$, there exists a constant $C_{\theta,\delta} > 0$ such that for any small $\varepsilon > 0$, we have

$$\left|\left\langle \left(w_\varepsilon - (w_{out} + \widetilde{w_\varepsilon})\right), \phi\right\rangle\right| \leq C_{\theta,\delta}\left(\frac{1}{T_0^{d/2-1}} + \frac{1}{T_0} + \alpha_\varepsilon^2 + \varepsilon\right).$$

This result roughly asserts that w_ε is asymptotic to $w_{out} + \widetilde{w_\varepsilon}$ as $\varepsilon \to 0$, up to carefully choosing the various parameters T_0, T_1, etc. Hence the proof of item (iii) of our main result essentially reduces to proving that $\widetilde{w_\varepsilon} \sim L_\varepsilon$ as $\varepsilon \to 0$.

Proof of Proposition 1.6.
The proof is obtained by gathering the statements of Proposition 1.7, Proposition 1.8, Proposition 1.9, Proposition 1.10 below. ■

The remainder part of this paragraph is devoted to a brief idea of proof of the above auxiliary Propositions that lead to Proposition 1.6.

- **Contribution of very small times $0 \leq t \leq T_0\,\varepsilon$.**

The contribution of very small times to $\langle w_\varepsilon, \phi\rangle = i\,\varepsilon^{-1}\displaystyle\int_0^{+\infty} e^{-\alpha_\varepsilon t}\,\langle U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon\rangle\, dt$, is

$$\frac{i}{\varepsilon}\int_0^{2T_0\varepsilon} \chi\left(\frac{t}{T_0\varepsilon}\right) e^{-\alpha_\varepsilon t}\langle U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon\rangle\, dt.$$

It is the main contribution to w_ε, provided T_0 is large enough. Indeed, we have the following fact, whose proof is based on a simple weak convergence argument.

Proposition 1.7. (See [4]). *Let $n^2(x)$ be any bounded and continuous refraction index. Then, if S and ϕ belong to $\mathcal{S}(\mathbb{R}^d)$, we have*

(i) For all time $T_0 > 0$,

$$\frac{i}{\varepsilon}\int_0^{2T_0\varepsilon} \chi\left(\frac{t}{T_0\varepsilon}\right) e^{-\alpha_\varepsilon t}\big\langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\big\rangle\, dt \underset{\varepsilon\to 0}{\longrightarrow} i\int_0^{2T_0} \chi\left(\frac{t}{T_0}\right)\left\langle \exp\left(it\left(\frac{\Delta_x}{2}+n^2(0)\right)\right)S, \phi\right\rangle dt.$$

(ii) There exists $C_d > 0$ which only depends on the dimension such that

$$\left|\left(\frac{i}{\varepsilon}\int_0^{2T_0\varepsilon} \chi\left(\frac{t}{T_0}\right)\big\langle \exp(it(\Delta_x/2+n^2(0)))S, \phi\big\rangle\, dt\right) - \langle w_{out}, \phi\rangle\right| \leq \frac{C_d}{T_0^{d/2-1}}.$$

• **Contribution of small, up to large times, away from the zero-energy level.**

The contribution to $\langle w_\varepsilon, \phi\rangle = i\,\varepsilon^{-1}\displaystyle\int_0^{+\infty} e^{-\alpha_\varepsilon t}\big\langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\big\rangle\, dt$ that is associated with small, up to large times, away from the zero-energy level, is

$$\frac{i}{\varepsilon}\int_{T_0\varepsilon}^{+\infty} e^{-\alpha_\varepsilon t}(1-\chi)\left(\frac{t}{T_0\varepsilon}\right)\big\langle(1-\chi_\delta)(H_\varepsilon)U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\big\rangle\, dt.$$

It is seen to be small, using a non-stationary phase argument in time, see [4] (this is the reason for the previous cut-off close to the initial time $t = 0$, where integrations by parts in time are forbidden). Indeed, we have the

Proposition 1.8. (See [4]). *Let n^2 be any long-range refraction index. Let S and ϕ belong to $L^2(\mathbb{R}^d)$. Then there exists a constant $C_\delta > 0$, which only depends on $\delta > 0$, such that for any small $\varepsilon > 0$ and any $T_0 > 0$, we have*

$$\left|\frac{1}{\varepsilon}\int_{T_0\varepsilon}^{+\infty}(1-\chi)\left(\frac{t}{T_0\varepsilon}\right)\big\langle\,(1-\chi_\delta(H_\varepsilon))\,U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon(t)\big\rangle\, dt\right| \leq C_\delta\left(\frac{1}{T_0}+\alpha_\varepsilon^2\right).$$

• **Contribution of large times, near the zero-energy level.**

The contribution to $\langle w_\varepsilon, \phi\rangle = i\,\varepsilon^{-1}\displaystyle\int_0^{+\infty} e^{-\alpha_\varepsilon t}\big\langle U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\big\rangle\, dt$ that is associated with large times, close to the zero-energy level, is

$$\frac{i}{\varepsilon}\int_{T_1}^{+\infty} e^{-\alpha_\varepsilon t}\,\langle\chi_\delta(H_\varepsilon)U_\varepsilon(t)S_\varepsilon, \phi_\varepsilon\rangle\, dt.$$

It is seen to be of order $O(\varepsilon^N)$, for all $N \in \mathbb{N}$, see [4]. Indeed, the semiclassical support of $\chi_\delta(H_\varepsilon)U_\varepsilon(t)S_\varepsilon$ goes to infinity in the x direction at speed of the order 1 (*i.e.* the semi-classical support lies in a region that is at distance of order t from the origin – this uses an argument due to Wang, see [18]), while the semi-classical support of ϕ_ε remains close to the origin. This argument relies on the fact that for T_1 large enough, the semiclassical supports of the two functions are disconnected, which in turn uses the non-trapping behaviour of the refraction index. We arrive at

Proposition 1.9. (See [4]). *Let n^2 be* any *long-range refraction index that is non-trapping. Let S and ϕ be in $\mathcal{S}(\mathbb{R}^d)$. Then there exist $\delta_0 > 0$ and $T_1(\delta_0) > 0$ such that for all time $T_1 \geq T_1(\delta_0)$ and any $0 < \delta < \delta_0$, there exists a constant C_δ such that*

$$\left| \frac{1}{\varepsilon} \int_{T_1}^{+\infty} e^{-\alpha_\varepsilon t} \big\langle \chi_\delta(H_\varepsilon) U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon \big\rangle \, dt \right| \quad \leq \quad C_\delta \varepsilon.$$

• **Contribution of small times near the zero-energy level**

The contribution to $\langle w_\varepsilon, \phi \rangle = i\,\varepsilon^{-1} \displaystyle\int_0^{+\infty} e^{-\alpha_\varepsilon t} \big\langle U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon \big\rangle \, dt$ that is associated with small times, close to the zero-energy level, is

$$\frac{i}{\varepsilon} \int_{T_0 \varepsilon}^{\theta} e^{-\alpha_\varepsilon t} \, (1 - \chi) \left(\frac{t}{T_0 \varepsilon} \right) \chi \left(\frac{t}{\theta} \right) \, \big\langle U_\varepsilon(t) \chi_\delta(H_\varepsilon) S_\varepsilon, \phi_\varepsilon \big\rangle dt.$$

Unlike in the previous case, the semiclassical supports of $U_\varepsilon(t)\chi_\delta(H_\varepsilon)S_\varepsilon$ and ϕ_ε *may* intersect for these values of time t. The whole point in [4] lies, roughly speaking, in proving a *dispersion estimate*. The key is to prove that the variable coefficients Schrödinger propagator $U_\varepsilon(t)$ has the same dispersive properties than the free Schrödinger propagator, corresponding to the case when $n^2 \equiv 0$, at least for small values of t such that $0 \leq t \leq \theta$ (for later times, the semiclassical support of $U_\varepsilon(t)\,S_\varepsilon$ is close to the classical trajectories $(X(t), \Xi(t))$, trajectories which in turn may come back close to the origin and contradict any dispersion effect). Indeed, for small times, the trajectory $(X(t), \Xi(t))$ is close to its first order expansion in time, which is the key to obtaining dispersive effects similar to the one at hand in the free case. Technically speaking, the proof relies on establishing that the propagator $U_\varepsilon(t)$ behaves like the free Schrödinger propagator for small times, a propagator whose symbol is $\exp(it|\xi|^2/\varepsilon)$, and which in turn has size $(\varepsilon/t)^{d/2}$ thanks to a stationary phase argument.

To obtain the desired statement, a wave packet approach is actually introduced, which strongly uses the work by Combescure and Robert ([7]). It allows to compute explicitly the propagator $U_\varepsilon(t)\,S_\varepsilon$, using the Hamiltonian flow and related, linearized, quantities, to obtain a representation of the form

$$\begin{aligned} &\frac{i}{\varepsilon} \int_{T_0 \varepsilon}^{\theta} e^{-\alpha_\varepsilon t} \, (1 - \chi) \left(\frac{t}{T_0 \varepsilon} \right) \chi \left(\frac{t}{\theta} \right) \, \big\langle U_\varepsilon(t) \chi_\delta(H_\varepsilon) S_\varepsilon, \phi_\varepsilon \big\rangle \, dt \\ &\qquad = \frac{1}{\varepsilon^{(5d+2)/2}} \int_{T_0 \varepsilon}^{\theta} \int_{\mathbb{R}^{6d}} e^{\frac{i}{\varepsilon}\,\psi(t,X)} \, a_N(t, X) \, dt \, dX + O_{\theta,\delta} \left(\varepsilon^N \right), \end{aligned} \tag{25}$$

where $X = (q, p, x, y, \xi, \eta) \in \mathbb{R}^{6d}$, where N is a possibly large integer, and the remainder term $O_{\theta,\delta}\left(\varepsilon^N\right)$ is upper bounded by $C_{\theta,\delta}\,\varepsilon^N$ for some $C_{\theta,\delta} > 0$ independent of ε, which depends on the chosen $\theta > 0$ and $\delta > 0$. Note that the amplitude a_N is defined in (38) below, while the *complex* phase function ψ is defined in (37) below. We refer to section 3 for details about the representation formula (25), which is a key ingredient in our proof of the main theorem.

With this representation at hand, we arrive at the

Proposition 1.10. (See [4]). *Let n^2 be* any *long-range potential which is non-trapping. For θ and δ small enough, there exists $C_\theta > 0$ and $C_{\theta,\delta} > 0$ such that for all $\varepsilon \leq 1$ we have*

$$\frac{1}{\varepsilon} \int_{T_0 \varepsilon}^{\theta} \chi \left(\frac{t}{\theta} \right) \left(1 - \chi \left(\frac{t}{T_0 \varepsilon} \right) \right) e^{-\alpha_\varepsilon t} \, \langle U_\varepsilon(t) \chi_\delta(H_\varepsilon) S_\varepsilon, \phi_\varepsilon \rangle \, dt \leq \frac{C_\theta}{T_0^{d/2-1}} + C_{\theta,\delta}\, \varepsilon. \tag{26}$$

2 Properties of the refraction index

2.1 Non-trapping behaviour

The goal of this subsection is to prove item (i) of our main Theorem 1.5.

We prove that the chosen refraction index $n^2(x) = n_\infty^2 - \lambda f(r) g(\theta_1)$ in (21) is *non-trapping* on the zero-energy level $H_0 = \{(x,\xi) \in \mathbb{R}^{2d},\ \text{s.t. } \xi^2/2 = n^2(x)\}$.

We first observe that the zero energy level has the more explicit value

$$H_0 = \left\{(x,\xi) \in \mathbb{R}^{2d},\ \text{s.t. } x = (r, \theta_1, \dots, \theta_{d-1}),\ \frac{\xi^2}{2} = n_\infty^2 - \lambda f(r) g(\theta_1)\right\}.$$

We readily define the following two regions. The first one is usually called the classically forbidden region: any trajectory living on the zero-energy level cannot reach the set $B_\emptyset$. The second one is sometimes called here the bump of the refraction index: it is the region where the refraction index actually *varies* with x. Outside this region, the refraction index is constant and the Hamiltonian trajectories associated with $h(x,\xi) = |\xi|^2/2 + n^2(x)$ are straight lines.

Definition 2.1. *(i) We denote by $B_\emptyset$ the set (classically forbidden region)*

$$B_\emptyset := \left\{x \in \mathbb{R}^d,\ \textit{s.t. } n^2(x) < 0\right\} = \left\{x = (r, \theta_1, \dots, \theta_{d-1}),\ \textit{s.t. } n_\infty^2 < \lambda f(r) g(\theta_1)\right\}.$$

(ii) We denote by B_p the set (bump)

$$B_p := \{x = (r, \theta_1, , \dots, \theta_{d-1}),\ \textit{s.t. } R-1 \le r \le R+1,\ |\theta_1| \le 2\theta_0\}.$$

Remark. *From the definition of $B_\emptyset$ and the two functions $f(r) = \chi(2(r-R))$ and $g(\theta_1) = \chi(\theta_1/\theta_0)$ it is clear that there exists $\mu \in]1,2[$ such that*

$$B_\emptyset \subset \left\{R - \frac{\mu}{2} \le r \le R + \frac{\mu}{2},\ |\theta_1| \le \mu\theta_0\right\}. \tag{27}$$

It suffices to take μ such that

$$0 < \chi(\mu) < \frac{n_\infty}{\sqrt{\lambda}} \ \textit{(hence } \mu \in]1,2[\textit{)}. \tag{28}$$

Our main step lies in proving the following escape estimate

Lemma 2.2. *Select the refraction index $n^2(x)$ as in (21) and assume condition (23) is fulfilled, namely $1 - \cos(2\theta_0) < 1/(2R)$. Take a Hamiltonian trajectory $X(t,x,\xi) \equiv X(t)$ living on the zero-energy level and define $x_0 := (R, 0, \dots, 0)$ in Cartesian coordinates.*

Then, there exists $\alpha > 0$, as well as $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, such that

$$\forall t \ge 0, \quad |X(t) - x_0|^2 \ge \alpha t^2 + \beta t + \gamma.$$

An immediate corollary of the above Lemma is

Corollary 2.3. *Assume condition (23) is fulfilled, namely $1 - \cos(2\theta_0) < 1/(2R)$. Then the refraction index $n^2(x)$ in (21) is non-trapping on the zero-energy level.*

Proof of Corollary 2.3.

Apply the preceding lemma and let $t \to +\infty$. ■

Proof of Lemma 2.2.

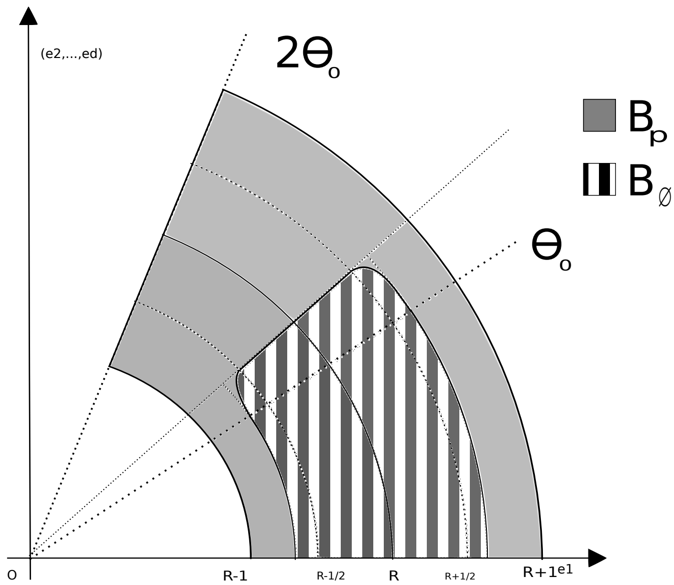

Figure 3: Bump of refraction index, and classically forbiden region

• **First step.** We compute the second derivative of $|X(t)-x_0|^2$ and get

$$
\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2}\left|X(t)-x_0\right|^2 &= \left\langle \frac{d^2}{dt^2}X(t), X(t)-x_0 \right\rangle + \left|\frac{dX}{dt}(t)\right|^2, \\
&= \left\langle \nabla n^2(X(t)), X(t)-x_0 \right\rangle + \left|\frac{dX}{dt}(t)\right|^2, \\
&= \left\langle \nabla n^2(X(t)), X(t)-x_0 \right\rangle + n^2(X(t)),
\end{aligned} \tag{29}
$$

where we have used the fact that the Hamiltonian trajectory $(X(t),\Xi(t))$ belongs to H_0. Letting $X(t) = r\,\vec{u}_r$ in hyperspherical coordinates and $x_0 = (R, 0, \dots, 0)$ in Cartesian coordinates, we obtain on the other hand

$$
\begin{aligned}
\left\langle \nabla n^2(X(t)), X(t)-x_0 \right\rangle &= \left\langle -\lambda f'(r) g(\theta_1)\vec{u}_r - \lambda \frac{f(r)}{r} g'(\theta_1)\vec{u}_{\theta_1}, r\,\vec{u}_r - R\,\vec{e}_1 \right\rangle, \\
&= F_r(r,\theta_1) + F_\theta(r,\theta_1),
\end{aligned}
$$

where

$$
F_r(r,\theta_1) = -\lambda f'(r) g(\theta_1)\left(r - R\cos(\theta_1)\right), \qquad F_\theta(r,\theta_1) = -\lambda \frac{R}{r} f(r) g'(\theta_1)\sin(\theta_1). \tag{30}
$$

Eventually we have

$$
\frac{1}{2}\frac{d^2}{dt^2}\left|X(t)-x_0\right|^2 = F_r(r,\theta_1) + F_\theta(r,\theta_1) + n^2(X(t)). \tag{31}
$$

Therefore, the lemma is proved once we establish the existence of $\alpha > 0$ such that

$$
F_r(x) + F_\theta(x) + n^2(x) \geq \alpha > 0
$$

whenever $x \in \Pi_x H_0 = \mathbb{R}^d \setminus B_\emptyset$ (where Π_x denotes the projection $(x, \xi) \mapsto x$ from $\mathbb{R}^{2d}$ to $\mathbb{R}^d$).

We readily notice that n^2 and F_θ are clearly non-negative function on the whole of $\mathbb{R}^d$.

• **Step two: non-negativity of F_r.** First, on $\mathbb{R}^d \setminus B_p$, the function F_r is zero, hence non-negative. In the same way on $B_p \cap \{R - 1/2 \le r \le R + 1/2\}$, we have $f' \equiv 0$, hence $F_r \equiv 0 \ge 0$. There remains to study the non-negativity of F_r on the two sets $\{R - 1 \le r \le R - 1/2,\ |\theta_1| \le 2\theta_0\}$ and $\{R + 1/2 \le r \le R + 1,\ |\theta_1| \le 2\theta_0\}$.

On $\{R - 1 \le r \le R - 1/2,\ |\theta_1| \le 2\theta_0\}$, we have

$$r - R\cos(\theta_1) \le R - \frac{1}{2} - R\cos(2\theta_0) = R(1 - \cos(2\theta_0)) - \frac{1}{2} < 0,$$

thanks to our assumption (23). Since $f' \ge 0$ on $\{R - 1 \le r \le R - 1/2\}$, we get $F_r \ge 0$ on $\{R - 1 \le r \le R - 1/2,\ |\theta_1| \le 2\theta_0\}$. A similar computation proves that $F_r \ge 0$ on the set $\{R + 1/2 \le r \le R + 1,\ |\theta_1| \le 2\theta_0\}$.

We have obtained that $F_r \ge 0$ on the whole of $\mathbb{R}^d$.

• **Step three: decomposition of $\mathbb{R}^d$.** We have just proved that $F_r(x) + F_\theta(x) + n^2(x) \ge 0$ for all $x \in \mathbb{R}^d$. We now wish to obtain a positive lower bound for $x \notin B_\emptyset$. The argument relies on the fact that the refraction index n^2 is positive away from the boundary $\partial B_\emptyset$, where $\partial B_\emptyset := \{(r, \theta_1, \ldots, \theta_{d-1}),\quad f(r)g(\theta_1) = n_\infty^2/\lambda\}$, while the term $F_r + F_\theta$ stemming from the gradient of the refraction index in (31) is positive close to the boundary $\partial B_\emptyset$. This is the reason for the decomposition we now introduce.

We define the set (piece of ring)

$$C_{\alpha,\beta} := \{R - \alpha \le r \le R + \alpha,\ -\beta \le \theta_1 \le \beta\}.$$

We know from the remark after Definition 2.1 that there exist $\mu \in]1, 2[$ such that

$$B_\emptyset \subset C_{R+\mu/2,\mu\theta_0}.$$

We therefore decompose

$$\mathbb{R}^d \setminus B_\emptyset = \left(\mathbb{R}^d \setminus C_{R+\mu/2,\mu\theta_0}\right) \cup \left(C_{R+\mu/2,\mu\theta_0} \setminus B_\emptyset\right).$$

We readily observe that, by construction of μ (namely $\chi(\mu)^2 \in]0, n_\infty^2/\lambda[$ – see (28)), for any $x \in \mathbb{R}^d \setminus C_{R+\mu/2,\mu\theta_0}$, we have the lower bound

$$n^2(x) = n_\infty^2 - \lambda f(r)g(\theta_1) \ge n_\infty^2 - \lambda\chi(\mu)^2 =: c_{n^2} > 0,$$

There only remains to prove the existence of $c_\nabla > 0$ such that $F_r + F_\theta \ge c_\nabla$ on $C_{R+\mu/2,\mu\theta_0} \setminus B_\emptyset$.

• **Step four: positive lower bound for $F_r + F_\theta$ on $C_{R+\mu/2,\mu\theta_0} \setminus B_\emptyset$.** Take $\nu \in]1, 2[$ such that

$$\frac{n_\infty}{\sqrt{\lambda}} < \chi(\nu) < 1.$$

where χ is the truncation function defined in (17). With this choice of ν, we clearly have, whenever $x \in C_{R+\nu/2,\nu\theta_0}$, the relation $n^2(x) = n_\infty^2 - \lambda\chi(2(r - R))\chi(\theta_1/\theta_0) \le n_\infty^2 - \lambda\chi(\nu)^2 < 0$, hence

$$C_{R+\nu/2,\nu\theta_0} \subset B_\emptyset \subset C_{R+\mu/2,\mu\theta_0}.$$

Therefore, it is enough to obtain a lower bound on $F_r + F_\theta$ on the set $C_{R+\mu/2,\mu\theta_0} \setminus C_{R+\nu/2,\nu\theta_0}$.

To this end, we decompose (see Figure 4)

$$\begin{aligned}
&C_{R+\mu/2,\mu\theta_0} \setminus C_{R+\nu/2,\nu\theta_0} \subset Z_r^1 \cup Z_r^2 \cup Z_\theta^1 \cup Z_\theta^2, \text{ with}\\
&Z_r^1 := \{R-\mu/2 \le r \le R-\nu/2,\ |\theta_1| \le \nu\theta_0\},\\
&Z_r^2 := \{R+\nu/2 \le r \le R+\mu/2,\ |\theta_1| \le \nu\theta_0\},\\
&Z_\theta^1 := \{R-\mu/2 \le r \le R+\mu/2,\ -\mu\theta_0 \le \theta_1 \le -\nu\theta_0\},\\
&Z_\theta^2 := \{R-\mu/2 \le r \le R+\mu/2,\ \nu\theta_0 \le \theta_1 \le \mu\theta_0\}.
\end{aligned}$$

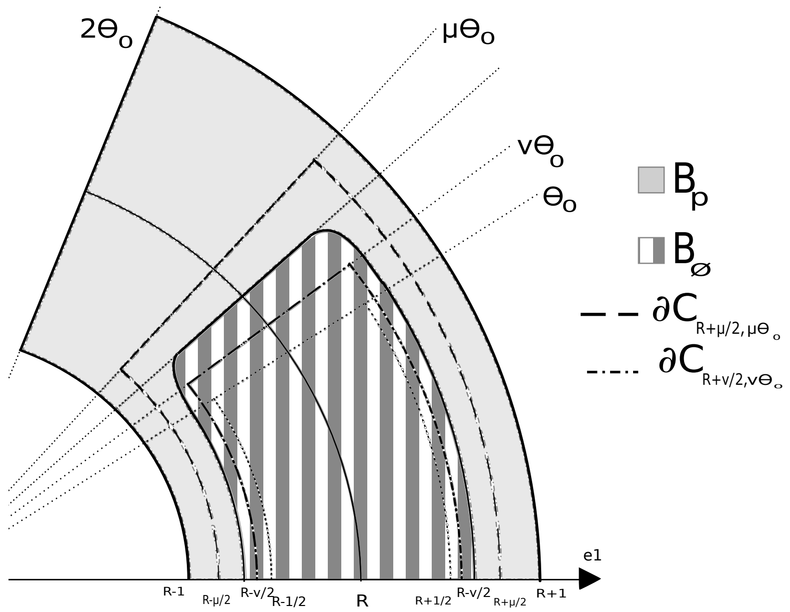

Figure 4: Zone of study

<u>On Z_r^1.</u> We use the structural hypothesis (23) to get

$$\begin{aligned}
F_r(x) &= -\lambda f'(r) g(\theta_1)(r - R\cos(\theta_1)) \ge -\lambda f'(r) g(\theta_1)(R - \frac{\nu}{2} - R\cos(2\theta_0))\\
&\ge \lambda f'(r) g(\theta_1)\frac{\nu-1}{2} \ge \lambda(\nu-1)\left(\min_{s\in[-\mu,-\nu]} \chi'(s)\right)\left(\min_{|s|\le\nu} \chi(s)\right) =: c_1 > 0. \qquad (32)
\end{aligned}$$

A similar proof establishes that, whenever $x \in Z_r^2$ we have

$$F_r(x) \ge \lambda(\nu-1)\left(\min_{s\in[\nu,\mu]} [-\chi'(s)]\right)\left(\min_{|s|\le\nu} \chi(s)\right) =: c_2 > 0.$$

<u>On Z_θ^1.</u> The important term is now F_θ. We have

$$\begin{aligned}
F_\theta(x) &= -\lambda \frac{R}{r} f(r) g'(\theta_1)\sin(\theta_1) \ge \lambda\frac{R}{r} f(r) g'(\theta_1)\sin(\nu\theta_0)\\
&\ge \lambda \frac{R}{\theta_0(R+\mu/2)}\left(\min_{|s|\le\mu}\chi(s)\right)\left(\min_{s\in[-\mu,-\nu]}\chi'(s)\right) =: c_3 > 0.
\end{aligned}$$

A similar argument establishes that, whenever $x \in Z_\theta^2$ we have

$$F_\theta(x) \geq \lambda \frac{R}{\theta_0(R+\mu/2)} \left(\min_{|s|\leq\mu} \chi(s)\right) \left(\min_{s\in[\nu,\mu]} [-\chi'(s)]\right) =: c_4 > 0.$$

Gathering all estimates, there exists a positive constant $c_\nabla > 0$ such that

$$\forall\, x \in C_{R+\mu/2,\mu\theta_0} \setminus C_{R+\nu/2,\nu\theta_0}, \qquad F_r(x) + F_\theta(x) \geq c_\nabla > 0.$$

- **Step five: end of the proof.** Putting all estimates together, we obtain

$$\forall\, x \in \Pi_x H_0 = \mathbb{R}^d \setminus B_\emptyset, \quad F_r(x) + F_\theta(x) + n^2(x) \geq \min(c_{n^2}, c_\nabla) =: \alpha > 0.$$

The lemma is proved. ∎

2.2 Refocusing Set

The goal of this subsection is to establish part (ii) of our main Theorem 1.5.

Our main result is

Proposition 2.4. *Let n^2 be the potential defined in (21). Assume the structural hypothesis (23) is fulfilled, namely $1 - \cos(2\theta_0) < 1/(2R)$. Then, the refocusing set defined in Definition 1.2 as*

$$M = \left\{(t,\xi,\eta) \in]0,+\infty[\times\mathbb{R}^{2d} \ \ s.t. \ \ \frac{|\eta|^2}{2} = n^2(0),\ X(t,0,\xi) = 0,\ \Xi(t,0,\xi) = \eta\right\}$$

satisfies

$$M = \left\{(T_R,\xi,\eta),\ \ s.t.\ \xi = -\eta = (r,\theta_1,\ldots,\theta_{d-1}),\ r = \sqrt{2n^2(0)},\ |\theta_1| \leq \theta_0\right\},$$

where $T_R > 0$ is the unique positive time such that $X(T_R, 0, (\sqrt{2n^2(0)}, 0\ldots,0)) = 0$.

Proof of Proposition 2.4.
Consider a trajectory $X(t,0,\xi) \equiv X(t)$ on the zero energy level, with $\xi = (r,\theta_1,\ldots,\theta_{d-1})$ in hyperspherical coordinates.

If $|\theta_1| \geq 2\theta_0$, it is clear that $X(t)$ is a straight line which never enters B_p, and the equation $X(t,0,\xi) = 0$ with $t > 0$ has no solution.

We need to understand the geometry when the trajectory reaches B_p, *i.e.* when $|\theta_1| < 2\theta_0$.

We prove below that two cases occur. If $|\theta_1| \leq \theta_0$, the trajectory remains along a line, and it is reflected by the refraction index towards the origin. If $\theta_0 < |\theta_1| < 2\theta_0$, the force acting on the trajectory has a non-vanishing component in the orthoradial direction, which prevents the trajectory to go back to the origin. The proposition follows.

Let us come to a proof.

- **First case:** $|\theta_1| \leq \theta_0$.

Consider the trajectory $Y(t)$ defined in hyperspherical coordinates as

$$Y(t) = (r(t),\theta_1,\ldots,\theta_{d-1}),$$

with $r(t)$ solution to the ordinary equation $r'' = -\lambda f'(r)$ with initial data

$$r(0) = 0, \qquad r'(0) = \sqrt{2n^2(0)}.$$

Then, $(Y(t), Y'(t))$ satisfies the Hamiltonian ODE (13) associated with $h(x,\xi) = |\xi|^2/2 + n^2(x)$. Since $Y(0) = X(0) = 0$, and $Y'(0) = X'(0) = \xi$, uniqueness provides $X(t) = Y(t)$ for all t. The trajectory $X(t)$ is radial.

It is clear that the radial trajectory $t \mapsto r(t)$ reaches the region $\{R-1 \le r \le R+1\}$ at time $t_e = (R-1)/|\xi| = (R-1)/\sqrt{2\,n^2(0)} > 0$, where $t_e = \inf\{t > 0,\ X(t) \in B_p\}$. Now, according to Corollary 2.3, the trajectory $r(t)$ necessarily leaves the region $\{R-1 \le r \le R+1\}$ at some later time $t_s > t_e$, where $t_s = \inf\{t > t_e,\ X(t) \notin B_p\}$. The trajectory can either leave the bump at $r = R-1$ or at $r = R+1$. The case $r = R+1$ is forbidden, for in the contrary case, using continuity, there would exist a time t_c such that $r(t_c) = R$, hence $X(t_c) \in B_\emptyset$, which is not allowed. Therefore, the trajectory leaves the bump B_p at $X(t_s)$ where $|X(t_s)| = r(t_s) = R-1$. Energy conservation, together with the fact that the trajectory is radial, implies that $X'(t_s) = -\xi$. Therefore, the trajectory for later times $t \ge t_s$ is a straight line with constant speed $-\xi$. We deduce that there exists a unique $T_R > t_s$ such that $X(T_R, 0, \xi) = 0$, and we have as desired $\Xi(T_R, 0, \xi) = -\xi$.

• **Second case:** $\theta_0 < |\theta_1| < 2\theta_0$**.**

We first assume that $d = 2$, and next generalize the argument to $d \ge 3$ using the symmetries of the system. To fix the ideas, we assume in the following that $\theta_0 < \theta_1 < 2\theta_0$, the proof being the same when θ_1 has the opposite sign.

$*$ **In dimension** $d = 2$**.**

Let $t_e = (R-1)/|\xi|$ be the time when the trajectory enters B_p, as in the preceding case.

On the one hand, since the velocity $\Xi(t_e)$ is radial and satisfies $\Xi(t_e) = |\xi|\,\vec{u}_r$, there is an $\varepsilon > 0$ such that $R-1 < |X(t)| < R+1$ whenever $t \in]t_e, t_e+\varepsilon]$. On the other hand, by assumption we have $\theta_1(t_e) = \theta_1 \in]\theta_0, 2\theta_0[$, and continuity implies there is an $\varepsilon > 0$ such that $\theta_0 < \theta_1(t) < 2\theta_0$ whenever $t \in [t_e, t_e+\varepsilon]$. Hence we may define

$$t_s := \sup\{t \ge t_e, \ \text{ s.t. } \ \forall t' \in [t_e, t], \quad \theta_1(t') \in]\theta_0, 2\theta_0[\quad \text{and} \quad X(t') \ne 0.\}.$$

Now, Hamilton's equations of motion (13) can be written in polar coordinates as

$$\begin{cases} r'' - r(\theta_1')^2 = -\lambda f'(r) g(\theta_1), \\ 2r'\theta_1' + r\theta_1'' = -\lambda \frac{f(r)}{r} g'(\theta_1). \end{cases}$$

Examining the second equation, we have $(r^2\theta_1')' = 2rr'\theta_1' + r^2\theta_1'' = -\lambda f(r) g(\theta_1)$, and we get whenever $r(t) \ne 0$,

$$\theta_1'(t) = -\frac{\lambda}{r^2(t)} \int_{t_e}^{t} f(r(s)) g'(\theta_1(s)) ds. \tag{33}$$

Therefore, since $f(r) \ge 0$ for any $r \ge 0$ while $f(r) > 0$ whenever $R-1 < r < R+1$, and since $g'(\theta_1) \le 0$ when $\theta_0 \le \theta_1 \le 2\theta_0$, while $g'(\theta_1) < 0$ when $\theta_0 < \theta_1 < 2\theta_0$ we get, with the above definitions and observations,

$$\theta_1'(t) > 0 \quad \forall t \in]t_e, t_s].$$

With this observation at hand, two cases may occur.

If $t_s = +\infty$, there is nothing to prove, for by definition of t_s, we have $X(t) \ne 0$ whenever $0 < t \le t_s = +\infty$.

In the case $t_s < +\infty$, we already know $X(t) \ne 0$ whenever $0 < t \le t_s$. Besides, since $\theta_1'(t) > 0$ whenever $0 < t \le t_s$, it is clear that the case $X(t_s) = 0$ is impossible (for in that case the trajectory would be a straight line passing through the origin on some interval $[t_*, t_s]$, in contradiction with $\theta_1'(t) > 0$ on $[t_*, t_s]$), hence $\theta_1(t_s) = 2\theta_0$ and $\theta_1'(t_s) > 0$. For that reason, the trajectory $X(t)$ for times $t > t_s$ is a straight line with constant velocity, which lies entirely in the set $2\theta_0 < \theta_1 < 2\theta_0 + \pi$. In particular, since $\theta_1'(t_s) > 0$, the trajectory cannot be radial and we have $X(t) \neq 0$ whenever $t > t_s$ in that case. This concludes the proof.

$*$ **In dimension** $d \geq 3$.

We use the invariance of n^2 under the action of $\mathbb{O}_{d,1}(\mathbb{R})$.

Take $\xi \in \mathbb{R}^d$ such that $|\xi| = \sqrt{2n^2(0)}$. Write $\xi = (\sqrt{2n^2(0)}, \theta_1, \dots, \theta_{d-1})$ in hyperspherical coordinates. There exists a matrix $A_\xi \in \mathbb{O}_{d,1}(\mathbb{R})$ such that $A_\xi\, \xi = (\sqrt{2n^2(0)}, \theta_1, 0, \dots, 0)$. On the other hand, denote by $(r(t), \theta_1(t))$ the solution of Hamilton's equations of motion (13) with initial data $(\sqrt{2n^2(0)}, \theta_1)$ in dimension 2. We set $Y(t) = A_\xi^{-1}(r(t), \theta_1(t), 0 \dots, 0)$. Then $Y(t)$ satisfies Hamilton's equations of motion (13), with initial data $Y(0) = 0$, $Y'(0) = \xi$. Uniqueness provides $Y(t) = X(t)$ for any $t > 0$. This, combined with the previous step, provides $X(t) \neq 0$ for any $t > 0$. ■

3 Convergence proof

The goal of this section is to prove item (iii) of our main Theorem 1.5.

The proof is performed in a number of steps. We begin by defining some necessary notation.

3.1 The linearized hamiltonian flow

Let $\varphi(t,x,\xi) = (X(t,x,\xi), \Xi(t,x,\xi))$ denote the flow associated with Hamilton's equations of motion (13). The linearized flow, written $F(t,x,\xi)$ below, is

$$F(t,x,\xi) = \frac{D\varphi(t,x,\xi)}{D(x,\xi)} := \begin{pmatrix} A(t,x,\xi) & B(t,x,\xi) \\ C(t,x,\xi) & D(t,x,\xi) \end{pmatrix},$$

where $A(t)$, $B(t)$, $C(t)$, $D(t)$ are by definition

$$A(t,x,\xi) = \frac{DX(t,x,\xi)}{Dx}, \quad B(t,x,\xi) = \frac{DX(t,x,\xi)}{D\xi},$$
$$C(t,x,\xi) = \frac{D\Xi(t,x,\xi)}{Dx}, \quad D(t,x,\xi) = \frac{D\Xi(t,x,\xi)}{D\xi}.$$

The linearisation of (13) leads to

$$\begin{cases} \dfrac{\partial}{\partial t} A(t,x,\xi) = C(t,x,\xi), & A(0,x,\xi) = Id, \\ \dfrac{\partial}{\partial t} C(t,x,\xi) = \dfrac{D^2 n^2}{Dx^2}(X(t,x,\xi)) A(t,x,\xi), & C(0,x,\xi) = 0, \end{cases} \tag{34}$$

as well as

$$\begin{cases} \dfrac{\partial}{\partial t} B(t,x,\xi) = D(t,x,\xi), & B(0,x,\xi) = Id, \\ \dfrac{\partial}{\partial t} D(t,x,\xi) = \dfrac{D^2 n^2}{Dx^2}(X(t,x,\xi)) B(t,x,\xi), & D(0,x,\xi) = 0. \end{cases} \tag{35}$$

Finally, we define for later purposes the matrix $\Gamma(t,x,\xi)$ as

$$\Gamma(t,x,\xi) = (C(t,x,\xi) + i\, D(t,x,\xi)) \,.\, (A(t,x,\xi) + iB(t,x,\xi))^{-1}. \tag{36}$$

3.2 A wave packet approach: preparing for a stationary phase argument

The intermediate result in Proposition 1.6 establishes roughly that $\langle w_\varepsilon, \phi\rangle \sim \langle w_{out} + \widetilde{w_\varepsilon}, \phi\rangle$ as $\varepsilon \to 0$. Therefore, item (iii) of our main Theorem reduces to proving $\langle \widetilde{w_\varepsilon}, \phi\rangle \sim \langle L_\varepsilon, \phi\rangle$ as $\varepsilon \to 0$.

Therefore, this preliminary paragraph is devoted to express the quantity

$$\langle \widetilde{w_\varepsilon}, \phi\rangle = \frac{1}{\varepsilon}\int_\theta^{T_1} \left(1-\chi\left(\frac{t}{\theta}\right)\right) e^{-\alpha_\varepsilon t} \langle U_\varepsilon(t)\chi_\delta(H_\varepsilon)S_\varepsilon, \phi_\varepsilon\rangle \, dt.$$

as an appropriate oscillatory integral. Our approach uses the technique developped in [4], which in turn strongly uses a wave packet theorem due to M. Combescure and D. Robert (see [7]). We skip here the details of the proof, referring to [4].

The main result in this paragraph is the following

Proposition 3.1. (See [7]) *Whenever $X = (q, p, x, \xi, y, \eta) \in \mathbb{R}^{6d}$ and $t \in \mathbb{R}$, define the complex phase*

$$\psi(t, X) := \int_0^t \left(\frac{p_s^2}{2} + n^2(q_s)\right) ds - p.(x-q) + p_t.(y-q_t)$$
$$+ x.\xi - y.\eta + i\frac{(x-q)^2}{2} + \frac{1}{2}\Gamma_t(y-q_t).(y-q_t), \tag{37}$$

where $q_t := X(t, q, p)$, $p_t := \Xi(t, q, p)$, and $\Gamma_t := \Gamma(t, q, p)$. Select an integer $N \in \mathbb{N}$. Select two truncation functions $\chi_0(q, p)$ and $\chi_1(x, y)$ both lying in $C_0^\infty(\mathbb{R}^{2d})$, and such that

$$\begin{aligned}
&supp\ \chi_0(q,p) \subset \{|q| \le 2\delta\} \cup \left\{||p|^2/2 - n^2(q)| \le 2\delta\right\}, \\
&\chi_0(q,p) \equiv 1 \ \ on\ \ \{|q| \le 3\delta/2\} \cup \left\{||p|^2/2 - n^2(q)| \le 3\delta/2\right\}, \\
&\chi_1(x,y) \equiv 1 \ \ close\ to\ \ (0,0).
\end{aligned}$$

Define the amplitude

$$a_N(t, X) := e^{-\alpha_\varepsilon t}(1-\chi)\left(\frac{t}{\theta}\right)\widehat{S}(\xi)\widehat{\phi}^*(\eta)\chi_0(q,p)\chi_1(x,y)P_N\left(t, q, p, \frac{y-q_t}{\sqrt{\varepsilon}}\right), \tag{38}$$

where $P_N(t, q, p, z)$ satisfies

$$P_N(t, q, p, x) := \frac{1}{\pi^{d/4}} det(A(t,q,p) + iB(t,q,p))_c^{-1/2} \mathcal{Q}_N(t, q, p, x), \tag{39}$$

and the square root $det(A(t,q,p) + iB(t,q,p))_c^{-1/2}$ is defined by continuously following the argument of the relevant complex number, starting from the value $det(A(0,q,p) + iB(0,q,p) = 1$ at time $t = 0$, while $\mathcal{Q}_N(t, q, p, x)$ is a polynomial in the variable $x \in \mathbb{R}^d$, whose coefficients vary smoothly with (t, q, p), and ε, and which satisfies

$$\mathcal{Q}_N(t, q, p, x) = 1 + O(\sqrt{\varepsilon})$$

in the relevant topology. More precisely, we have

$$\begin{cases} \mathcal{Q}_N(t, q, p, x) = 1 + \displaystyle\sum_{(k,j)\in I_N} \varepsilon^{\frac{k}{2}-j} p_{k,j}(t, q, p, x), \\ I_N = \{1 \le j \le 2N-1, 1 \le k-2j \le 2N-1,\ k \ge 3j\}, \end{cases} \tag{40}$$

where each $p_{k,j}$ has at most degree k in the variable x.

Then, the following holds

$$\langle \widetilde{w_\varepsilon}, \phi \rangle = \frac{1}{\varepsilon^{(5d+2)/2}} \int_\theta^{T_1} \int_{\mathbb{R}^{6d}} e^{\frac{i}{\varepsilon}\psi(t,X)} a_N(t,X) dt dX + O_{T_1,\delta}(\varepsilon^N). \tag{41}$$

Sketch of proof of Proposition 3.1.
Using the short-hand notation $\widetilde{\chi}_\delta(t) := e^{-\alpha_\varepsilon t}(1-\chi)(t/\theta)$, we have

$$\langle \widetilde{w_\varepsilon}, \phi \rangle = i/\varepsilon \int_\theta^{T_1} \widetilde{\chi}_\delta(t) \, \langle \chi_\delta(H_\varepsilon) S_\varepsilon, U_\varepsilon(-t)\phi_\varepsilon \rangle \, dt. \tag{42}$$

To compute the term $U_\varepsilon(-t)\phi_\varepsilon$ accurately, we use a projection over the overcomplete basis of $L^2(\mathbb{R}^d)$ obtained by using the so-called gaussian wave-packets, namely the family of functions indexed by $(q,p) \in \mathbb{R}^{2d}$ defined by

$$\varphi^\varepsilon_{q,p}(x,\xi) := \frac{1}{(\pi\varepsilon)^{d/4}} \exp\left(\frac{i}{\varepsilon} p. \left(x - \frac{q}{2}\right)\right) \exp\left(-\frac{(x-q)^2}{2\varepsilon}\right).$$

The point indeed is that, as proved by Combescure and Robert in [7], we have

$$\begin{aligned} U_\varepsilon(-t)\, \varphi^\varepsilon_{q,p}(x,\xi) &= O_{T_1,\delta}(\varepsilon^N) + \\ &\frac{1}{\varepsilon^{d/4}} \exp\left(\frac{i}{\varepsilon} p_t. \left(x - \frac{q_t}{2}\right)\right) \exp\left(-\frac{|x-q_t|^2}{2\varepsilon}\right) \\ &\exp\left(\frac{i}{\varepsilon}\left[\int_0^t \left(\frac{p_s^2}{2} + n^2(q_s)\right) ds - \frac{q_t \cdot p_t - q \cdot p}{2}\right]\right) P_N\left(t, q, p, \frac{x - q_t}{\sqrt{\varepsilon}}\right) \end{aligned} \tag{43}$$

in $L^\infty([0,T_1]; L^2(\mathbb{R}^d))$. In other words, we have a quite explicit complex-phase/amplitude representation of the Schrödinger propagator when acting on the gaussian wave packets.

This observation leads to writing, successively, in (42)

$$\begin{aligned} \langle \chi_\delta(H_\varepsilon) S_\varepsilon, U_\varepsilon(-t)\phi_\varepsilon \rangle &= \frac{1}{(2\pi\varepsilon)^d} \int_{\mathbb{R}^{2d}} \left\langle \chi_\delta(H_\varepsilon) S_\varepsilon, \varphi^\varepsilon_{q,p} \right\rangle \left\langle \varphi^\varepsilon_{q,p}, U_\varepsilon(-t)\phi_\varepsilon \right\rangle dq\, dp\,, \\ &= \frac{1}{(2\pi\varepsilon)^d} \int_{\mathbb{R}^{2d}} \left\langle \chi_\delta(H_\varepsilon) S_\varepsilon, \varphi^\varepsilon_{q,p} \right\rangle \left\langle U_\varepsilon(t)\varphi^\varepsilon_{q,p}, \phi_\varepsilon \right\rangle dq\, dp\,. \end{aligned}$$

Now, the idea is to replace the factor $U_\varepsilon(t)\varphi^\varepsilon_{q,p}$ by its approximation derived above. Yet a few preliminary steps are in order. The first one uses the truncation in energy $\chi_\delta(H_\varepsilon)$, together with the functional calculus for pseudo-differential operators of Helffer and Robert (see [11]), to replace this truncation by an explicit truncation near the set $p^2/2 + n^2(q) = 0$, up to small error terms. The second step consists in using the Parseval formula to write (we want to exploit the source term S_ε on the Fourier side)

$$\langle S_\varepsilon\,, \phi^\varepsilon_{q,p} \rangle = \frac{1}{(2\pi\varepsilon)^{d/2}} \int e^{i\frac{x\cdot\xi}{\varepsilon}} \, \widehat{S}(\xi)\, \phi^\varepsilon_{q,p}(x)\, dx\, d\xi = \frac{1}{(2\pi\varepsilon)^{d/2}} \int \widetilde{\chi}(x)\, e^{i\frac{x\cdot\xi}{\varepsilon}} \, \widehat{S}(\xi)\, \phi^\varepsilon_{q,p}(x)\, dx\, d\xi,$$

for some function $\widetilde{\chi}(x)$ that truncates close to $x = 0$, and similarly

$$\left\langle U_\varepsilon(t)\varphi^\varepsilon_{q,p}, \phi_\varepsilon \right\rangle = \frac{1}{(2\pi\varepsilon)^{d/2}} \int \widetilde{\chi}(y)\, e^{i\frac{y\cdot\eta}{\varepsilon}} \, \widehat{\phi}(\eta) \, \left(U_\varepsilon(t)\varphi^\varepsilon_{q,p}\right)(y)\, dy\, d\eta,$$

These two steps explain the truncation factors χ_0 and χ_1 in the Proposition, which act close to the zero energy-level in phase-space (this is where functional calculus is used) and close to the origin in physical space. The last step consists in exploiting formula (43) in the obtained representation.

Eventually, one obtains the desired formula. ■

3.3 Preparing for a stationary phase argument

This slightly technical paragraph is devoted to proving that the obtained phase ψ in Proposition 3.1 satisfies the assumptions of the stationary phase Theorem.

Our main result in this paragraph is the Proposition after the following Lemma.

Lemma 3.2. *Let n^2 be any smooth refraction index. Then, the following holds*

(i) The stationary set associated with the phase ψ in (37), defined as

$$M_X := \left\{(t, X) = (t, q, p, x, \xi, y, \eta) \in [\theta, T_1] \times \mathbb{R}^{6d} \ \text{ s.t. } \ \nabla_{t,X}\psi(t, X) = 0 \ \text{ and } \operatorname{Im}\psi(t, X) = 0\right\}$$

satisfies

$$M_X = \{(t, q, p, x, \xi, y, \eta) \ \text{ s.t. } \ x = y = q = 0,\ \xi = p,\ (t, p, \eta) \in M\}, \tag{44}$$

where we recall that $M = \{(t, p, \eta),\ X(t, 0, p) = 0,\ \Xi(t, 0, p) = \eta,\ \eta^2/2 = n^2(0)\}$ by definition.

(ii) We have, whenever $m = (t, X) \in M_X$, the relation

$$\begin{aligned}\operatorname{Ker}(D^2\psi_{|_m}) = \big\{&(T, Q, P, X, \Xi, Y, H) \in]0, +\infty[\times\mathbb{R}^{6d},\ X = Y = Q = 0, \\ &\Xi = P,\ \eta^T H = 0, B_t(0, p)\, P + T\eta = 0,\ -H + D_t(0, p)\, P + T\nabla n^2(0) = 0\big\}.\end{aligned} \tag{45}$$

Note that this Lemma does not use the particular structure of our index.

Proof of Lemma 3.2. A mere computation of $\operatorname{Im}\psi$ and $\nabla\psi$ allows to write (44). Differentiating $\nabla\psi$ once allows to write (45). For more details, the reader may check [4]. ■

With this Lemma at hand, our key result in this section is the following

Proposition 3.3. *Let n^2 be the refraction index defined in (21). We recall that the refocusing set M is computed in Lemma 2.4 and satisfies*

$$M = \left\{(T_R, \xi, \eta) \ \text{ s.t. } \ \xi = -\eta = (r, \theta_1, \dots, \theta_{d-1}),\ r = \sqrt{2n^2(0)},\ |\theta_1| \le \theta_0\right\}.$$

Now, take any

$$\begin{aligned} m \in \mathring{M}_X = \Big\{&(t, q, p, x, \xi, y, \eta) \ \text{ s.t. } \ x = y = q = 0,\ \xi = p, \\ &(t, p, \eta) \in M,\ \text{with}\, p = (r, \theta_1, \dots, \theta_{d-1}), \ \text{ and } \ |\theta_1| < \theta_0\Big\}\end{aligned}$$

Then, we have

$$\operatorname{Ker} D^2\psi|_m = T_m M_X,$$

where $T_m M_X$ denotes the space tangent to M_x at point m.

The remainder part of this subsection is devoted to the proof of Proposition 3.3. We begin by proving the Proposition in the case

$$m = m_0 := (T_R, 0, p_0, 0, p_0, 0, -p_0), \quad \text{where} \quad p_0 := (\sqrt{2n^2(0)}, 0, \dots, 0).$$

We next generalize the result to other values of m, using the symmetries of the problem.

3.3.1 Proof of Proposition 3.3 when $m = m_0$

The computation of $T_{m_0}M_X$ on the one hand is rather easy

Lemma 3.4. *The space $T_{m_0}M_X$ is given by*

$$T_{m_0}M_X = \{(T, Q, P, X, \Xi, Y, H) \ s.t. \ X = Y = Q = T = 0, \ \Xi = P = -H, \ P.p_0 = 0\}.$$

Proof of Lemma 3.4. This is a mere computation starting from the definition of the refocusing set M, as $M = \{(t, p, \eta), \ X(t, 0, p) = 0, \ \Xi(t, 0, p) = \eta, \ \eta^2/2 = n^2(0)\}$. ■

In order to determine Ker $D^2\psi_{|m_0}$ the first step it to compute the matrices B_t and D_t involved in the linearized flow, see (3.3.2).

Lemma 3.5. *Let n^2 be the potential defined in (21). Then, we have*

$$D(T_R, 0, p_0) := \frac{\partial \Xi}{\partial \xi}(T_R, 0, p_0) = -I_d, \quad B(T_R, 0, p_0) := \frac{\partial X}{\partial \xi}(T_R, 0, p_0) = \begin{pmatrix} b_{11} & 0 \\ 0 & O_{d-1} \end{pmatrix}, \quad (46)$$

where I_d is the identity matrix, $b_{11} \in \mathbb{R}$ and O_{d-1} is a square matrix of dimension $d-1$ equal to 0.

Proof of Lemma 3.5.

We consider $x_0(t, 0, p) = (x_0^1(t, 0, p), \ldots, x_0^d(t, 0, p))$ the solution to (13) with initial data $x_0(0, 0, p) = 0$ and $x_0'(0, 0, p) = p$.

We recall that the index n^2 is invariant under the action of $\mathbb{O}_{d,1}(\mathbb{R}^d)$. Thus we first compute the components of D and B that are invariant under $\mathbb{O}_{d,1}(\mathbb{R}^d)$, namely their first column. We next compute the other columns by using the symmetries again, in conjunction with a perturbation argument.

- **Computation of $\frac{\partial \Xi}{\partial \xi_1}(T_R, 0, p_0)$ and $\frac{\partial X}{\partial \xi_1}(T_R, 0, p_0)$**

We start with $\frac{\partial \Xi_j}{\partial \xi_1}(T_R, 0, p_0)$ for $j \geq 2$. We have

$$\frac{\partial \Xi_j}{\partial \xi_1}(T_R, 0, p_0) = \lim_{\varepsilon \to 0} \frac{\Xi_j\left(T_R, 0, (\sqrt{2n^2(0)} + \varepsilon, 0 \ldots, 0)\right) - \Xi_j\left(T_R, 0, (\sqrt{2n^2(0)}, 0 \ldots, 0)\right)}{\varepsilon}.$$

Since the trajectory is radial we have

$$\Xi_j\left(T_R, 0, (\sqrt{2n^2(0)} + \varepsilon, 0 \ldots, 0)\right) = \Xi_j\left(T_R, 0, (\sqrt{2n^2(0)}, 0 \ldots, 0)\right) = 0, \quad \forall \, j \geq 2.$$

Hence, $\frac{\partial \Xi_j}{\partial \xi_1}(T_R, 0, p_0) = 0$, $\forall \, j \geq 2$. A similar argument provides $\frac{\partial X_j}{\partial \xi_1}(T_R, 0, p_0) = 0$, $\forall \, j \geq 2$. There remains to determine the first coefficient of D, namely $\frac{\partial \Xi_1}{\partial \xi_1}(T_R, 0, p_0)$. Since the trajectory is radial, and by conservation of the energy, we have for ε small enough

$$\Xi_1\left(T_R, 0, (\sqrt{2n^2(0)} + \varepsilon, 0, \ldots, 0)\right) = -\left(\sqrt{2n^2(0)} + \varepsilon\right),$$
$$\Xi_1\left(T_R, 0, (\sqrt{2n^2(0)}, 0, \ldots, 0)\right) = -\sqrt{2n^2(0)}.$$

Thus,

$$d_{11} := \lim_{\varepsilon \to 0^+} \frac{\Xi\left(T_R, 0, (\sqrt{2n^2(0)} + \varepsilon, 0, \ldots, 0)\right) - \Xi\left(T_R, 0, (\sqrt{2n^2(0)}, 0, \ldots, 0)\right)}{\varepsilon} = -1.$$

• **Computation of** $\frac{\partial \Xi(T_R,0,p_0)}{\partial \xi_j}$ **and** $\frac{\partial X(T_R,0,p_0)}{\partial \xi_j}$ $(j \geq 2)$

Considering the symmetries of the problem, it is enough to consider the case $j = 2$: the other components may be determined using the same argument.

We perturb the initial speed along the direction e_2, by a factor ε (see Figure 5).

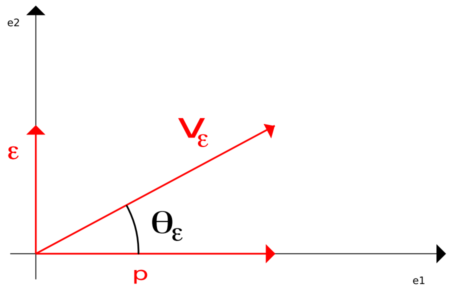

Figure 5: Perturbation of the initial speed

Let $X_\varepsilon(t)$ be the solution of the perturbed problem

$$\Big\{ X_\varepsilon''(t) = \nabla n^2(X_\varepsilon(t)), \qquad X_\varepsilon(0) = 0, \qquad X_\varepsilon'(0) = p_0 + \varepsilon\, e_2.$$

We expand $X_\varepsilon(t)$ with respect to ε and obtain $X_\varepsilon(t) = X_0(t) + \varepsilon X_1(t) + \ldots$. With this notation we have $X_1(t) = \frac{\partial X}{\partial \xi_2}(t)$ and $X_1'(t) = \frac{\partial \Xi}{\partial \xi_2}(t)$. To obtain the expansion in ε, we go back to the previous case ($j = 1$) using a change of variables. Indeed, for ε small enough, the trajectory is radial along the direction $X_\varepsilon'(0)$. Let $(\widetilde{e_1}, \ldots, \widetilde{e_d})$ be a new basis defined by $\widetilde{e_j} := O_\varepsilon e_j$, with

$$O_\varepsilon := \begin{pmatrix} \cos(\theta_\varepsilon) & -\sin(\theta_\varepsilon) & 0 & \ldots & 0 \\ \sin(\theta_\varepsilon) & \cos(\theta_\varepsilon) & 0 & \ldots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & I_{d-2} & \\ 0 & 0 & & & \end{pmatrix}, \qquad \cos(\theta_\varepsilon) = \frac{p_0}{p_0^2 + \varepsilon^2}, \qquad \sin(\theta_\varepsilon) = \frac{\varepsilon}{p_0^2 + \varepsilon^2}.$$

Let $\widetilde{X_\varepsilon}$ be the coordinates of X_ε in $(\widetilde{e_1}, \ldots, \widetilde{e_d})$. Since $O_\varepsilon^{-1} \nabla n^2(X_\varepsilon) = \nabla n^2(\widetilde{X_\varepsilon})$, we clearly have

$$\widetilde{X_\varepsilon}''(t) = \nabla n^2(\widetilde{X_\varepsilon}(t)), \quad \widetilde{X_\varepsilon}(0) = 0, \quad \widetilde{X_\varepsilon}'(0) = (\sqrt{\varepsilon^2 + p_0^2}, 0, \ldots, 0) = p_0 + O(\varepsilon^2).$$

Hence it is clear that $\widetilde{X_\varepsilon}(t) = \widetilde{X_0}(t) + O(\varepsilon^2)$. Therefore, we recover

$$X_0(t) + \varepsilon X_1(t) = O_\varepsilon \left(\widetilde{X_0}(t) + O(\varepsilon^2) \right) = (I_d + \varepsilon E + O(\varepsilon^2))(\widetilde{X_0}(t) + O(\varepsilon^2)),$$

with

$$E := \begin{pmatrix} 0 & -\frac{1}{p_0} & 0 & \ldots & 0 \\ \frac{1}{p_0} & 0 & 0 & \ldots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & I_{d-2} & \\ 0 & 0 & & & \end{pmatrix}.$$

In other words, we have

$$\forall t \in \mathbb{R}, \qquad X_0(t) = \widetilde{X_0}(t) \qquad \text{and} \qquad X_1(t) = E\widetilde{X_0}(t).$$

Since the Hamiltonian trajectory goes back to the origin at time T_R, we deduce

$$\frac{\partial X}{\partial \xi_2}(T_R, 0, p_0) = X_1(T_R) = E\widetilde{X_0}(T_R, 0, p_0) = E \times 0 = 0.$$

In the same way, we have

$$\begin{aligned}\frac{\partial \Xi}{\partial \xi_2}(T_R, 0, p_0) = X_1'(T_R) = E\widetilde{X_0}'(T_R) &= {}^t\left(-\frac{x'^2_{0,R}(T_R)}{p_0}, \frac{x'^1_{0,R}(T_R)}{p_0}, x'^3_{0,R}(T_R), \dots, x'^d_{0,R}(T_R)\right), \\ &= {}^t(0, 1, 0, \dots, 0).\end{aligned}$$

The columns of B and D (for $j \geq 3$) are determined in the similar way. This leads to (46). ■

At this stage, we deduce the

Corollary 3.6. *$Ker\, D^2\psi_{|m_0} = T_{m_0}M_X$.*

Proof of Corollary 3.6.
According to (45), we have

$$\begin{aligned}Ker(D^2\psi_{|m}) = \{&(T, Q, P, X, \Xi, Y, H),\ X = Y = Q = 0, \\ &\Xi = P, \eta^T H = 0, B_{T_R}(0,p)\, P + T\eta = 0, -H + D_{T_R}(0,p)\, P + T\nabla n^2(0) = 0\}.\end{aligned}$$

Since $\eta = -p_0$, we recover $H = (0, H_2, \dots, H_d)$ (in Cartesian coordinates). Since $\nabla n^2(0) = 0$, we deduce that $D_{T_R}(0,p)\, P = H$. According to Lemma 3.5, we deduce that $H = -P$. Finally, $B_{T_R(0,p)}\, P = 0$ hence $T = 0$. Thus,

$$\text{Ker}\, D^2\psi_{|m_0} = \{(T, Q, P, X, \Xi, Y, H),\ X = Y = Q = T = 0,\ P = \Xi = -H,\ P.p_0 = 0\}.$$

Using Lemma 3.4, the proof is complete. ■

3.3.2 Proof of Proposition 3.3 for any m

In this subsection, we prove the

Lemma 3.7. *$\forall m \in \overset{\circ}{M}_X$, we have $T_m M_X = \text{Ker}\, D^2\psi_{|m}$.*

Proof of Lemma 3.7.
The idea is to use a family of transformations which leave $\overset{\circ}{M}_X$ and n^2 invariant (in a sense we define later), next to transport the equality $\text{Ker}\, D^2\psi_{|m_0} = T_{m_0}M_X$ to any $m \in \overset{\circ}{M}_X$.

Family of transformations. Let $m = (t, q, p, x, \xi, y, \eta) \in \overset{\circ}{M}_X$. We write $m = (T_R, 0, p, 0, p, 0, -p)$ for some $p \in \sqrt{2n^2(0)}\,\mathbb{S}^{d-1}$ Thus, there exists $R_p \in \mathbb{O}(\mathbb{R}^d)$ such that $R_p(p) = p_0$. We define the map $\widetilde{R}_m : \mathbb{R}^{6d+1} \longrightarrow \mathbb{R}^{6d+1}$ by

$$\widetilde{R}_m(t, q, p, x, \xi, y, \eta) = (t, R_p(q), R_p(y), R_p(x), R_p(\xi), R_p(y), R_p(\eta)). \tag{47}$$

By construction we have $\widetilde{R}_m\,(m) = m_0$.

Action on the tangent place. We have identified that the set M_X satisfies

$$M_X = \underbrace{\{(t,q,p,x,\xi,y,\eta),\ \text{s.t.}\ \ t = T_R,\ q = x = y = 0,\ p = \xi = -\eta,\ p^2/2 = n^2(0)\}}_{:=\widetilde{M_X}}$$
$$\cap\,\{p = (r,\theta_1,\ldots,\theta_{d-1})\ \text{ with } |\theta_1| \le \theta_0\}.$$

The set $\widetilde{M_X}$ is clearly invariant under the action of $\widetilde{R}_m$. Therefore, by restricting the domain in the variable θ_1, it is clear that whenever $m \in \overset{\circ}{M}_X$, there exists a neighbourhood U of m in $\overset{\circ}{M}_X$ such that $U_0 := \widetilde{R}_m U \subset \overset{\circ}{M}_X$. Since the application $\widetilde{R}_m$ is a linear map from U to U_0 which satisfies $\widetilde{R}_m\,(m) = m_0$, we deduce

$$\widetilde{R}_m\ (T_m M_X) = T_{m_0} M_X.$$

Action on the kernel. We now compute the set $\widetilde{R}_m(\text{Ker}(D^2\psi_{|_m}))$, as follows

$$\begin{aligned}
\widetilde{R}_m(\text{Ker}(D^2\psi_{|_m})) &= \{(T, R_p\,Q, R_p\,P, R_p\,X, R_p\,\Xi, R_p\,Y, R_p\,H),\ \text{s.t.}\ X = Y = Q = 0,\\
&\qquad p.H = 0,\ B_{T_R}(0,p)\,P + Tp = 0,\ D_{T_R}(0,p)\,P = H\},\\
&= \{(T,Q,P,X,\Xi,Y,H),\ \text{s.t.}\ X = Y = Q = 0,\\
&\qquad p.R_p^{-1}H = 0,\ B_{T_R}(0,p)\,R_p^{-1}P + Tp = 0,\ D_{T_R}(0,p)\,R_p^{-1}P = R_p^{-1}H\}.\\
&= \{(T,Q,P,X,\Xi,Y,H),\ \text{s.t.}\ X = Y = Q = 0,\\
&\qquad p_0.H = 0,\ R_pB_{T_R}(0,p)\,R_p^{-1}P + Tp_0 = 0,\ R_pD_{T_R}(0,p)\,R_p^{-1}P = H\}.
\end{aligned}$$

On the other hand, we claim that

$$R_pB_{T_R}(0,p)R_p^{-1} = B_{T_R}(0,p_0), \qquad R_pD_{T_R}(0,p)R_p^{-1} = B_{T_R}(0,p_0). \tag{48}$$

Assuming the above identity is proved, we immediately deduce

$$\widetilde{R}_m\ (\text{Ker}(D^2\psi_{|_m})) = \text{Ker}\,D^2\psi_{|_{m_0}}.$$

We conclude by writing

$$\widetilde{R}_m\ (\text{Ker}(D^2\psi_{|_m})) = \text{Ker}D^2\psi_{|_{m_0}} = T_{m_0}M_X = \widetilde{R}_m\ (T_m M_X).$$

Thus, there only remains to prove (48). By construction of the potential we clearly have

$$R_pX(t,0,p) = X(t,0,p_0),\ \text{ as well as } n^2\,(R_px)) = n^2\,(x)\,,$$

whenever $x/|x|$ lies in the angular sector $|\theta_1| \le \theta_0$. This provides

$$\frac{D^2n^2}{Dx^2}(X(t,0,p_0)) = \frac{D^2n^2}{Dx^2}(R_p\,X(t,0,p)) = R_p\,\frac{D^2n^2}{Dx^2}(X(t,0,p))\,R_p^{-1}.$$

Therefore, using the differential equation (3.3.2) relating the time evolution of B_t and D_t, we recover the following system

$$\begin{cases}
\dfrac{\partial}{\partial t}R_pB(t,0,p)R_p^{-1} = R_pD(t,0,p)R_p^{-1}, & R_pB(0,0,p)R_p^{-1} = Id,\\
\dfrac{\partial}{\partial t}R_pD(t,0,p)R_p^{-1} = R_p\dfrac{D^2n^2}{Dx^2}(X(t,0,p))B(t,0,p)R_p^{-1}, & \\
\qquad = \dfrac{D^2n^2}{Dx^2}(R_pX(t,0,p))R_pB(t,0,p)R_p^{-1} & \\
\qquad = \dfrac{D^2n^2}{Dx^2}(X(t,0,p_0))R_pB(t,0,p)R_p^{-1} & R_pD(0,0,p)R_p^{-1} = 0.
\end{cases}$$

Uniqueness of solutions to a differential system then gives

$$\forall t, \quad R_p B_t(0,p) R_p^{-1} = B_t(0,p_0), \quad R_p D_t(0,p) R_p^{-1} = D_t(0,p_0).$$

Relation (48) is proved. ■

3.3.3 A useful byproduct of the proof of Proposition 3.3

Lemma 3.8. *Let n^2 be the refraction index defined in (21). Take any $m \in M_X$, written as $m = (T_R, 0, p, 0, p, 0, -p)$ with $p = \left(\sqrt{2n^2(0)}, \theta_1, \theta_2, \ldots, \theta_{d-1}\right)$ according to Lemma 2.4. Then,*

$$\psi(m) \textit{ is constant on the set } |\theta_1| \leq \theta_0.$$

Proof of Lemma 3.8. Considering the actual value of $\psi(m)$, various terms need to be considered. The term $\int_0^t (p_s^2/2 + n^2(q_s))\,ds$ is clearly constant whenever $|\theta_1| \leq \theta_0$. The same statement holds for the factor $p_t \cdot q_t$. The only non-obvious factor is $\Gamma_t q_t \cdot q_t$. As in the preceding proof we write

$$\begin{aligned}\Gamma_t(0,p) q_t(0,p) \cdot q_t(0,p) &= \Gamma_t(0,p) q_t(0, R_p^{-1} p_0) \cdot q_t(0, R_p^{-1} p_0)\\ &= R_p \Gamma_t(0,p) R_p^{-1} q_t(0,p_0) \cdot q_t(0,p_0).\end{aligned}$$

There remains to write

$$\begin{aligned}R_p \Gamma_t(0,p) R_p^{-1} &= R_p \left(C_t(0,p) + i D_t(0,p)\right) \cdot \left(A_t(0,p) + i B_t(0,p)\right)^{-1} R_p^{-1}\\ &= \left(R_p C_t(0,p) R_p^{-1} + i R_p D_t(0,p) R_p^{-1}\right) \cdot \left(R_p A_t(0,p) R_p^{-1} + i R_p B_t(0,p) R_p^{-1}\right)^{-1}\\ &= \Gamma_t(0,p_0)\end{aligned}$$

for we already know that $R_p B_t(0,p) R_p^{-1} = B_t(0,p_0)$, $R_p D_t(0,p) R_p^{-1} = D_t(0,p_0)$, and a similar proof establishes $R_p A_t(0,p) R_p^{-1} = B_t(0,p_0)$, $R_p C_t(0,p) R_p^{-1} = D_t(0,p_0)$. ■

3.4 The stationary phase argument: Proof of item (iii) of our main Theorem

The main result of the present section is

Proposition 3.9. *Let n^2 be the potential constructed according to 21. Select a source $S \in \mathcal{S}(\mathbb{R}^d)$. Then, the following holds.*

(i) If $\operatorname{supp}\left(\widehat{S}(\xi)\right) \cap \partial I_{\theta_0} = \emptyset$*, we have*

$$\forall\, \phi \in \mathcal{S}(\mathbb{R}^d), \quad \langle \widetilde{w_\varepsilon} - L_\varepsilon, \phi \rangle = O_{T_1,\delta}(\sqrt{\varepsilon}),$$

where $\langle L_\varepsilon, \phi \rangle$ is defined in (24) above (see also the Remark after Theorem 1.5), and $\partial I_{\theta_0} = \{\xi = (|\xi|, \theta_1, \ldots, \theta_{d-1})$ such that $\theta_1 = \pm\theta_0\}$ (see definition 1.4).

(ii) In the general case we have

$$\forall\, \phi \in \mathcal{S}(\mathbb{R}^d), \quad \langle \widetilde{w_\varepsilon} - L_\varepsilon, \phi \rangle = o_{T_1,\delta}(\varepsilon^0).$$

Proof of Proposition 3.9. Due to the fact that the stationary set M_X in the to-be-developped stationary phase argument has a boundary at $\theta_1 = \pm\theta_0$, the argument is in two steps. This is the reason why the above Proposition distinguishes between two cases.

•• Proof of Proposition 3.9-part (i)

Outside the stationary set M_X associated with the complex phase ψ, the oscillatory integral (41) defining $\langle \widetilde{w_\varepsilon}, \phi\rangle$ is of order $O(\varepsilon^\infty)$. On the stationary set M_X and near the support of a_N, the stationary set M_X is a submanifold *without boundary*, having codimension $k = 6d+1-(d-1) = 5d+2$. Indeed, thanks to the hypothesis on the support of $\widehat{S}$, we have $\operatorname{supp} a_N \cap \partial M_X = \emptyset$.

Let us now come to the explicit application of the stationary phase Theorem to the oscillatory integral (41). Writing $p = (r, \theta_1, \ldots, \theta_{d-1})$ in hyperspherical coordinates, we define the application:

$$\begin{array}{rcl} \gamma : \mathbb{R}^{6d+1} \cap \text{supp } a_N & \longrightarrow & \mathbb{R}^{5d+2} \times \mathbb{S}^{d-1} \\ (t,q,p,x,\xi,y,\eta) & \longmapsto & (\underbrace{t-T_R, q, x, y, \xi - p, \eta + p, r - \sqrt{2n^2(0)}}_{=:\alpha}, \underbrace{\theta_1, \ldots, \theta_{d-1}}_{=:\theta}) \end{array}$$

The map γ is a C^∞-diffeomorphism between supp a_N and $\gamma\,(\text{supp } a_N)$. Furthermore, we have by construction

$$(t, X) \in M_X \cap \text{supp } a_N \Longleftrightarrow \alpha = 0.$$

The new coordinates (α, θ) are adapted to the stationary set M_X associated with ψ. Making the change of variables $(t, X) = \gamma^{-1}(\alpha, \theta)$ in the integral defining $\langle \widetilde{w_\varepsilon}, \phi\rangle$ we have

$$\begin{aligned} &\langle \widetilde{w_\varepsilon}, \phi\rangle = O_{\delta,T_1}(\varepsilon^N) + &(49)\\ &\frac{1}{\varepsilon^{(5d+2)/2}} \int_{\gamma(supp\ a_N)} e^{\frac{i}{\varepsilon}\psi\circ\gamma^{-1}(\alpha,\theta)} \left(\widehat{S}(.)\widehat{\phi}^*(.)P_N\left(.,.,.,\frac{.}{\sqrt{\varepsilon}}\right)\right) \circ \gamma^{-1}(\alpha,\theta)\chi_3(\alpha,\theta)\, r^{d-1}\, d\alpha\, d\sigma(\theta), \end{aligned}$$

where $d\sigma(\theta)$ denotes the standard euclidean surface measure on the unit sphere $\mathbb{S}^{d-1}$, and χ_3 is a truncation function on some compact set, a neighbourhood of M_X, whose precise value is irrelevant. Here we have used the *non-stationary* phase Theorem to reduce the original integral to an integral on a given compact set.

Since for all point $m \in M_X \cap \operatorname{supp} a_N$ we have $\operatorname{Ker}(D^2\psi_{|m}) = T_m M_X$ (Lemma 3.7), the function $D^2\psi$ is non-degenerate in the normal direction to M_X, which gives

$$\det\left(\frac{D^2\psi\circ\gamma^{-1}}{D\alpha^2}(0,\theta)\right) \neq 0.$$

Furthermore, the projection of $\gamma(\text{supp } a_N)$ onto the space variable θ is the angular sector

$$\Pi_\theta I_{\theta_0} := \{(\theta_1, \ldots, \theta_{d-1}), \quad \theta_1 \in]-\theta_0, \theta_0[\},$$

where Π_θ denotes the projection $(r, \theta_1, \ldots, \theta_{d-1}) \mapsto (\theta_1, \ldots, \theta_{d-1})$. We can now apply the stationary phase Theorem in (49). Remembering that the codimension of the stationary set M_X associated with ψ is $5d+2$, we obtain that for *any* integer L there exists a sequence $(Q_{2\ell}(\partial))_{\ell\in\{0,\ldots,L\}}$

of operators of order 2ℓ such that

$$
\begin{aligned}
&\langle \widetilde{w_\varepsilon}, \phi\rangle = \\
&C_1 \int_{\Pi_\theta I_{\theta_0}} \frac{\exp\left(i\frac{\pi}{4}\mathrm{sgn}\frac{D^2\psi\circ\gamma^{-1}}{D\alpha^2}(0,\theta)\right)}{\left|\det\frac{D^2\psi\circ\gamma^{-1}}{D\alpha^2}(0,\theta)\right|} \exp\left(\frac{i}{\varepsilon}\psi\circ\gamma^{-1}(0,\theta)\right) \\
&\qquad\qquad \left(\left(Q_0(.)\widehat{S}(.)\widehat{\phi^*}(.)P_N\left(.,.,.,\frac{.}{\sqrt{\varepsilon}}\right)\right)\circ\gamma^{-1}\chi_3\right)(0,\theta)\, d\sigma(\theta) \\
&+\int_{\Pi_\theta I_{\theta_0}} \exp\left(\frac{i}{\varepsilon}\psi\circ\gamma^{-1}(0,\theta)\right)\sum_{\ell=1}^{L}\varepsilon^\ell\, Q_{2\ell}(\partial)\left(\left(\widehat{S}(.)\widehat{\phi^*}(.)P_N\left(.,.,.,\frac{.}{\sqrt{\varepsilon}}\right)\right)\circ\gamma^{-1}\chi_3\right)(0,\theta)\, d\sigma(\theta) \\
&+O\left(\varepsilon^{L+1}\sup_{K\le 2L+d+3}\left\|\partial^K_{(\alpha,\theta)}\left(\widehat{S}(.)\widehat{\phi^*}(.)P_N\left(.,.,.,\frac{.}{\sqrt{\varepsilon}}\right)\circ\gamma^{-1}\chi_3\right)\right\|_{L^\infty}\right) + O_{\delta,T_1}(\varepsilon^N) \qquad (50)\\
&:= I_\varepsilon + II_\varepsilon + III_\varepsilon + O_{\delta,T_1}(\varepsilon^N),
\end{aligned}
$$

with the value

$$C_1 = (2\pi)^{(5d+2)/2}\,(2n^2(0))^{(d-1)/2}.$$

The last line in (50) serves as a definition of the three terms I_ε, II_ε and III_ε, and the L^∞-norm in III_ε is evaluated on a compact set of values of (α,θ), whose precise value is irrelevant.

We compute these three contributions. Note that the retained value of the integer L remains to be determined at this stage.

• **Contribution of the remainder term III_ε in (50).**

This term is best studied by coming back to the original variables (t,X) instead of (α,θ). Expanding the k-th order derivatives involved in this term, we clearly have

$$
\begin{aligned}
III_\varepsilon &= O\left(\varepsilon^{L+1}\sup_{K\le 2L+d+3}\left\|\partial^K_{(t,X)}\left(\widehat{S}(.)\widehat{\phi^*}(.)P_N\left(.,.,.,\frac{.}{\sqrt{\varepsilon}}\right)\right)\right\|_{L^\infty}\right)\\
&= O\left(\varepsilon^{L+1}\sup_{K\le 2L+d+3}\left\|\partial^K_{(t,X)}P_N\left(.,.,.,\frac{.}{\sqrt{\varepsilon}}\right)\right\|_{L^\infty}\right).
\end{aligned}
$$

Hence, since

$$P_N(t,q,p,x) = \pi^{-d/4}\det(A(t,q,p)+iB(t,q,p))_c^{-1/2}\mathcal{Q}_N(t,q,p,x),$$

we recover

$$III_\varepsilon = O\left(\varepsilon^{L+1}\sup_{K\le 2L+d+3}\left\|\partial^K_{(t,q,p,y)}\left(\mathcal{Q}_N(t,q,p,(y-q_t)/\sqrt{\varepsilon})\right)\right\|_{L^\infty}\right).$$

Lastly, using (40) we have

$$\mathcal{Q}_N(t,q,p,x) := 1 + \sum_{(k,j)\in I_N}\varepsilon^{\frac{k}{2}-j}p_{k,j}(t,q,p,x),$$

where $p_{k,j}$ has at most degree k in x. We deduce

$$
\begin{aligned}
III_\varepsilon &= \sum_{(k,j)\in I_N} O\left(\varepsilon^{\frac{k}{2}-j+L+1}\sup_{K\le 2L+d+3}\left\|\partial^K_{(t,q,p,y)}\left(p_{k,j}(t,q,p,(y-q_t)/\sqrt{\varepsilon})\right)\right\|\right)\\
&= \sum_{(k,j)\in I_N} O\left(\varepsilon^{\frac{k}{2}-j+L+1-\frac{k}{2}}\right) = O\left(\varepsilon^{L+1-(2N-1)}\right),
\end{aligned}
$$

where we have used that $j \leq 2N-1$ whenever $(k,j) \in I_N$ (see (40)). There remains to chose

$$L = 2N - 1$$

to recover

$$III_\varepsilon = O(\varepsilon).$$

• **Contribution of II_ε in (50).**
This estimate is more delicate. Firstly, we have

$$II_\varepsilon = \sum_{\ell=1}^{L} \varepsilon^\ell O\left(\left\| Q_{2\ell}(\partial) \left(\left(\widehat{S}(.)\widehat{\phi^*}(.) P_N\left(.,.,.,\frac{.}{\sqrt{\varepsilon}}\right)\right) \circ \gamma^{-1}\chi_3 \right)(0,\theta) \right\|_{L^\infty} \right).$$

Hence, going back to the (t,X) variables again, and remembering that the relation $(\alpha,\theta) = (0,\theta)$ implies $y = q_t = 0$ and $t = T_R$, we recover the identity

$$II_\varepsilon = \sum_{\ell=1}^{L} \varepsilon^\ell \, O\left(\sup_{K \leq 2\ell} \left\| \partial^K_{(t,q,p,y)} \Big|_{y=q_t=0, t=T_R} \left(P_N\left(t,q,p,\frac{y-q_t}{\sqrt{\varepsilon}}\right)\right) \right\|_{L^\infty} \right),$$

where the L^∞-norm is evaluated on some compact set of values of p. Now, inserting the exact value of P_N, we may write

$$\begin{aligned} II_\varepsilon &= \sum_{\ell=1}^{L} \varepsilon^\ell \, O\left(\sum_{(k,j)\in I_N} \sup_{K \leq 2\ell} \left\| \partial^K_{(t,q,p,y)} \Big|_{y=q_t=0, t=T_R} \left(\varepsilon^{\frac{k}{2}-j} \, p_{k,j}\left(t,q,p,\frac{y-q_t}{\sqrt{\varepsilon}}\right)\right) \right\|_{L^\infty} \right) \\ &= \sum_{\ell=1}^{L} \sum_{(k,j)\in I_N} \varepsilon^\ell \, \varepsilon^{\frac{k}{2}-j} \, O\left(\sup_{K \leq 2\ell} \left\| \partial^K_{(t,q,p,y)} \Big|_{y=q_t=0, t=T_R} \left(p_{k,j}\left(t,q,p,\frac{y-q_t}{\sqrt{\varepsilon}}\right)\right) \right\|_{L^\infty} \right) \end{aligned}$$

Hence, using the fact that each $p_{k,j}$ is a polynomial in its last argument, so that the above derivatives evaluated at $y = q_t = 0$ only leave the zero-th order term in the derived polynomial, we recover

$$\begin{aligned} II_\varepsilon &= O\left(\sum_{\ell=1}^{L} \sum_{(k,j)\in I_N} \varepsilon^\ell \, \varepsilon^{\frac{k}{2}-j} \sup_{K \leq 2\ell} \varepsilon^{-K/2} \right) \\ &= O\left(\sum_{\ell=1}^{L} \sum_{(k,j)\in I_N} \varepsilon^\ell \, \varepsilon^{\frac{k}{2}-j} \, \varepsilon^{-\ell} \right) = O\left(\sum_{\ell=1}^{L} \sum_{(k,j)\in I_N} \varepsilon^{\frac{k}{2}-j} \, \varepsilon^{-\ell} \right) = O\left(\varepsilon^{1/2}\right), \end{aligned}$$

where we have used that $k - 2j \geq 1$ whenever $(k,j) \in I_N$.

• **Contribution of I_ε in (50).**
The integral defining I_ε has the following more explicit value, where $p = (\sqrt{2n^2(0)}, \theta_1, \ldots, \theta_{d-1})$, namely

$$I_\varepsilon = C_1 \int_{\Pi_\theta I_{\theta_0}} \frac{e^{i\frac{\pi}{4}\mathrm{sgn}\left(\frac{D^2\psi\circ\gamma^{-1}}{D\alpha^2}(0,\theta)\right)}}{\det\left(\frac{D^2\psi\circ\gamma^{-1}}{D\alpha^2}(0,\theta)\right)} \exp(\left(\frac{i}{\varepsilon}\psi(T_R,0,p,0,p,0,-p)\right)$$
$$\det(A(T_R,0,p) + iB(T_R,0,p))_c^{-1/2} \, \widehat{S}(p) \, \widehat{\phi}^*(-p) \, d\theta_1 \ldots d\theta_{d-1},$$

On top of that, we have

$$\psi(T_R,0,p,0,p,0,-p)=\int_0^{T_R}\left(\frac{|p_s(0,p)|^2}{2}+n^2(q_s(0,p))\right)ds,$$

while the fact that n^2 is radial implies that $\psi(T_R,0,p,0,p,0,-p)=\psi(T_R,0,p_0,0,p_0,0,-p_0)$ whenever $p\in I_{\theta_0}$. For the same reason, we also have whenever $\theta\in\Pi_\theta I_{\theta_0}$ the relation

$$\frac{e^{i\frac{\pi}{4}\operatorname{sgn}\left(\frac{D^2\psi\circ\gamma^{-1}}{D\alpha^2}(0,\theta)\right)}}{\det\left(\frac{D^2\psi\circ\gamma^{-1}}{D\alpha^2}(0,\theta)\right)}=\frac{e^{i\frac{\pi}{4}\operatorname{sgn}\left(\frac{D^2\psi\circ\gamma^{-1}}{D\alpha^2}(0,0)\right)}}{\det\left(\frac{D^2\psi\circ\gamma^{-1}}{D\alpha^2}(0,0)\right)}$$

together with the identity, valid whenever $p\in I_{\theta_0}$,

$$\det(A(T_R,0,p)+iB(T_R,0,p))_c^{-1/2}=\det(A(T_R,0,p_0)+iB(T_R,0,p_0))_c^{-1/2}.$$

Eventually, we have obtained

$$I_\varepsilon=C_{n^2,d}\,e^{\left(\frac{i}{\varepsilon}\int_0^{T_R}\left(\frac{|p_s(0,p_0)|^2}{2}+n^2(q_s(0,p_0))\right)ds\right)}\int_{I_{\theta_0}}\widehat{S}(p)\widehat{\phi^*}(-p)\,d\sigma_{\theta_0}(p),\tag{51}$$

with

$$C_{T_R,d}:=\frac{(2\pi)^{5d+2}e^{i\frac{\pi}{4}\operatorname{sgn}\left(\frac{D^2\psi\circ\gamma^{-1}}{D\alpha^2}(0,0)\right)}}{\det\left(\frac{D^2\psi\circ\gamma^{-1}}{D\alpha^2}(0,0)\right)}\det(A(T_R,0,p_0)+iB(T_R,0,p_0))_c^{-1/2}.$$

This ends the proof of Proposition 3.9-part (i).

•• Proof of Proposition 3.9-part (ii)

In that case, the argument is essentially the same (a stationary phase argument in the variable α), up to a convenient use of the dominated convergence Theorem (to deal with the variable θ_1, and more specifically with the boundary $\theta_1=\pm\theta_0$).

Namely, we first write, as in the proof of part (i) of the Proposition,

$$\begin{aligned}\langle\widetilde{w_\varepsilon},\phi\rangle&=O_{\delta,T_1}(\varepsilon^N)+\\ &\frac{1}{\varepsilon^{(5d+2)/2}}\int_{\gamma(supp\ a_N)}e^{\frac{i}{\varepsilon}\psi\circ\gamma^{-1}(\alpha,\theta)}\left(\widehat{S}(.)\widehat{\phi}^*(.)P_N\left(.,.,.,\frac{.}{\sqrt{\varepsilon}}\right)\right)\circ\gamma^{-1}(\alpha,\theta)\chi_3(\alpha,\theta)\,r^{d-1}\,d\alpha\,d\sigma(\theta),\end{aligned}\tag{52}$$

where χ_3 is a truncation function on some compact set, a neighbourhood of M_X, whose precise value is irrelevant. Here we have used the non-stationary phase Theorem to reduce the original integral to an integral on a given compact set. The key point now lies in writing,

$$\begin{aligned}\langle\widetilde{w_\varepsilon},\phi\rangle&=O_{\delta,T_1}(\varepsilon^N)+\\ &\int d\sigma(\theta)\underbrace{\left(\frac{1}{\varepsilon^{(5d+2)/2}}\int d\alpha\,e^{\frac{i}{\varepsilon}\psi\circ\gamma^{-1}(\alpha,\theta)}\left(\widehat{S}(.)\widehat{\phi}^*(.)P_N\left(.,.,.,\frac{.}{\sqrt{\varepsilon}}\right)\right)\circ\gamma^{-1}(\alpha,\theta)\chi_3(\alpha,\theta)\,r^{d-1}\right)}_{=:J_\varepsilon(\theta)}.\end{aligned}\tag{53}$$

With this formulation in mind, our next objective is to prove that whenever $\eta>0$ is a small parameter we have

$$\int_{|\theta_1\pm\theta_0|\leq\eta}d\sigma(\theta)\ |J_\varepsilon(\theta)|\leq C\,\eta,\tag{54}$$

for some $C > 0$ independent of ε and η. It is clear indeed that the upper-bound (54), in conjunction with part (i) of the Proposition, provides a complete proof of Proposition 3.9-part (ii).

Let us now concentraate on the case $|\theta_1 - \theta_0| \leq \eta$ (the proof in the case $|\theta_1 + \theta_0| \leq \eta$ is the same).

In order to prove (54), we fix a value $(\theta_2^0, \dots, \theta_{d-1}^0)$ and we prove that, given $(\theta_2^0, \dots, \theta_{d-1}^0)$, there is an $\eta > 0$, and a $C > 0$ independent of ε, such that

$$\forall \theta \text{ such that } |\theta - (\theta_0.\theta_2^0, \dots, \theta_{d-1}^0)| \leq \eta, \qquad \text{we have } |J_\varepsilon(\theta)| \leq C. \tag{55}$$

Covering the whole set $\{\theta \in \mathbb{S}^{d-1}; |\theta_1 - \theta_0| \leq \eta\}$ by finitely many sets of the form $\{|\theta - (\theta_0.\theta_2^0, \dots, \theta_{d-1}^0)| \leq \eta\}$ clearly provides the desired relation (54) once (55) is proved.

Now, relation (55) results from an application of the stationary phase Theorem, with *complex phase* and *with parameter*. Here α is the variable used for the stationary phase itself, while θ is the parameter, and $\psi \circ \gamma^{-1}$ is the complex phase. We introduce the short-hand notation $\theta^0 = (\theta_0, (\theta')^0) = (\theta_0, \theta_2^0, \dots, \theta_{d-1}^0)$ for convenience. It has already been established[6] that

$$\begin{aligned}
&\operatorname{Im}\left(\psi \circ \gamma^{-1}\right)(\alpha, \theta) \geq 0, \quad \forall (\alpha, \theta), \\
&\operatorname{Im}\left(\psi \circ \gamma^{-1}\right)(\alpha = 0, \theta = \theta^0) = 0, \\
&\nabla_\alpha \left(\psi \circ \gamma^{-1}\right)(\alpha = 0, \theta = \theta^0) = 0, \\
&\det\left(\frac{D^2 \psi \circ \gamma^{-1}}{D\alpha^2}\right)(\alpha = 0, \theta = \theta^0) \neq 0
\end{aligned}$$

Therefore, the stationary phase theorem with parameter ensures that close to $\theta = \theta^0$ there is an expansion of the form

$$J_\varepsilon(\theta) = e^{i\phi(\theta)/\varepsilon}\left(\sum_{\ell=0}^{L} \varepsilon^\ell \left(Q_{2\ell}(\partial_\alpha)\left(\widehat{S}(.)\,\widehat{\phi^*}(.)\,P_N\left(.,.,\frac{.}{\sqrt{\varepsilon}}\right) \circ \gamma^{-1}\,\chi_3(.)\right)\right)^0(\theta)\right) + R(\varepsilon, L, \theta),$$

for some smooth functions ϕ and $R(\varepsilon, L, \theta)$, where the $Q_{2\ell}$'s are differential operators of order 2ℓ in the variable α, and, for any function $u(\alpha, \theta)$, the notation $u^0(\theta)$ refers to any smooth function $u^0(\theta)$ that belongs to the same residue class than the original function $u(\alpha, \theta)$ modulo the ideal generated by $\nabla_\alpha \psi \circ \gamma^{-1}(\alpha, \theta)$ (see Hörmander [12], sect. 7.7, for the details). With this notation, we actually have $\phi = \left(\psi \circ \gamma^{-1}\right)^0$. Besides, the remainder term R satisfies as the term III_ε in the previous step an estimate of the form

$$|R(\varepsilon, L, \theta)| \leq C_L\, \varepsilon^{L+1}\left(\sup_{K \leq 2(L+1)} \left\|\partial_\alpha^K\left(\left(\widehat{S}(.)\,\widehat{\phi^*}(.)\,P_N\left(.,.,.,\frac{.}{\sqrt{\varepsilon}}\right)\right) \circ \gamma^{-1}\,\chi_3(.)\right)\right\|_{L^\infty}\right),$$

for some constant $C_L > 0$ independent of ε, and provided θ is close to θ^0 (independently of ε). These two ingredients immediately provide, using the same estimates as we did for the terms III_ε and II_ε above, the upper-bound, valid for θ close to θ^0,

$$|J_\varepsilon(\theta)| \leq C\left(\sum_{\ell=0}^{L} \varepsilon^\ell \left(Q_{2\ell}(\partial_\alpha)\left(\widehat{S}(.)\,\widehat{\phi^*}(.)\,P_N\left(.,.,\frac{.}{\sqrt{\varepsilon}}\right) \circ \gamma^{-1}\,\chi_3(.)\right)\right)^0(\theta)\right) + R(\varepsilon, L, \theta),$$

Gathering powers of ε as in the previous part of the proof, provides the upper bound

$$|J_\varepsilon(\theta)| \leq C,$$

[6] *Stricto sensu*, these relations have only be proved when $|\theta_1| < \theta_0$, and we here extend the result to the case $\theta_1 = \theta_0$. This is allowed due to the invariance of the phase on the parameter θ whenever $|\theta_1| \leq \theta_0$ – Lemma 3.8.

where C does not depend on ε and θ is close to θ^0, independently of ε. Point (55) is proved.

We immediately deduce that (54) holds, and the proof of Proposition 3.9 – part (ii) is complete.

■

3.5 Conclusion

Gathering the intermediate result in Proposition 1.6, together with Proposition 3.9, gives item (iii) of Theorem 1.5, by conveniently choosing the parameters δ, θ, T_0 and T_1.

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